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Semigroups of locally Lipschitz operators associated with semilinear evolution equations

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Abstract

In this paper we introduce the notion of semigroups of locally Lipschitz operators which provide us with mild solutions to the Cauchy problem for semilinear evolution equations, and characterize such semigroups of locally Lipschitz operators. This notion of the semigroups is derived from the well-posedness concept of the initial-boundary value problem for differential equations whose solution operators are not quasi-contractive even in a local sense but locally Lipschitz continuous with respect to their initial data. The result obtained is applied to the initial-boundary value problem for the complex Ginzburg–Landau equation.

Keywords: Semigroup of locally Lipschitz operators; Semilinear evolution equation; Infinitesimal generator; Semilinear stability condition; Subtangential condition; Comparison function; Maximal solution

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1. Introduction

The notion of well-posedness of the abstract Cauchy problem for A_0, $u'(t) = A_0u(t)$ for $t \geqslant 0$, is closely related with the theory of semigroups of operators. In the linear case, it is known [6,21] that the abstract Cauchy problem for A_0 is well-posed in the sense that it has a unique solution depending continuously on initial data if and only if A_0 is the infinitesimal generator of a (C_0) semigroup, and the characterization of infinitesimal generators of (C_0) semigroups is known as the Hille–Yosida theorem [9,24]. In the nonlinear case, the generation theorem of contractive semigroups [1–3,11,16] plays an important role in showing that a given initial-boundary value problem has a unique solution in some sense and the solution operator is contractive. Unlike the linear case, the above-mentioned results cannot be applied to the initial-boundary value problem for certain differential equations whose solution operators are not quasi-contractive even in a local sense but locally Lipschitz continuous with respect to their initial data. This leads us to the notion of semigroups of locally Lipschitz operators. As a special case, the continuous infinitesimal generator of a semigroup of Lipschitz operators was characterized in [12] and the characterization was applied to the initial-boundary value problem for a quasilinear wave equation with dissipation. For wider class of applications, it is strongly desired to extend their results to the case where infinitesimal generators are not always continuous. This paper is devoted to the well-posedness of the Cauchy problem for the semilinear evolution equation

$$u'(t) = (A + B)u(t) \quad \text{for } t \geqslant 0 \tag{SP}$$

in a general Banach space X. Here A is the infinitesimal generator of a (C_0) semigroup on X and B is a nonlinear operator from a subset D of X into X.

Semilinear problems of the form (SP) arise in various fields of mathematical science. Studying such problems from the theoretic point of view is recognized to be important and a nonlinear continuous perturbation problem has been discussed by many authors [5,7,10,14,15,17,19,20, 23]. In those papers, the nonlinear semigroups generated by $A + B$ are quasi-contractive. We are interested in discussing a characterization of semigroups of locally Lipschitz operators on D which provide us with mild solutions to the Cauchy problem for (SP). The main theorem (Theorem 2.3) states roughly that such a semigroup is generated by $A + B$ if and only if a subtangential condition, a growth condition and a semilinear stability condition in terms of a family of metric-like functionals on $X \times X$ are satisfied. In Section 2, the main theorem is stated and the proof of the necessity part is given. In Section 3, we show the so-called local uniformity of the subtangential condition and apply the result to construct a sequence of approximate solutions to the Cauchy problem for (SP). Section 4 is devoted to the sufficiency part of the main theorem, namely, the generation of semigroups of locally Lipschitz operators associated with semilinear evolution equations. In the final Section 5, an application of the main theorem to the complex Ginzburg–Landau equation is discussed.

2. Assumptions and main theorem

Let X be a general Banach space with norm $\|\cdot\|$ and D a subset of X. We begin by listing up basic assumptions on A and B appearing in (SP).

(A) The operator A is the infinitesimal generator of a (C_0) semigroup $\{T(t);\ t \geqslant 0\}$ on X.

Throughout this paper, we use the notations $a \wedge b := \min(a, b)$, $a \vee b := \max(a, b)$, $\mathbb{R}_+ := [0, \infty)$ and $\overline{\mathbb{R}}_+ := [0, \infty]$.

In order to impose the local continuity for the nonlinear operator B from D into X, we employ a vector-valued functional $\varphi = (\varphi_i)_{i=1}^n$ on X to $\overline{\mathbb{R}}_+^n$ such that $D \subset D(\varphi) := \{x \in X;\ \varphi_i(x) < \infty$ for all $i = 1, 2, \ldots, n\}$, and the order '$\leqslant$' in $\mathbb{R}^n$ defined in the way that $\alpha = (\alpha_i)_{i=1}^n \leqslant \beta = (\beta_i)_{i=1}^n$ if and only if $\alpha_i \leqslant \beta_i$ for all $i = 1, 2, \ldots, n$.

(φ) For each $\alpha \in \mathbb{R}_+^n$, the level set $D_\alpha := \{v \in D;\ \varphi(v) \leqslant \alpha\}$ is closed in X.
(B) For each $\alpha \in \mathbb{R}_+^n$, the operator B is continuous on D_α in X.

The Cauchy problem for semilinear evolution equation (SP) with initial condition $u(0) = u_0$ is denoted by (SP; u_0). The Cauchy problem (SP; u_0) may not admit strong solutions. In this paper we employ the following notion of generalized solutions.

Definition 2.1. Let $u_0 \in D$ and $\tau > 0$. A continuous function $u(\cdot) : [0, \tau] \to X$ is called a *mild solution to* (SP; u_0) *on* $[0, \tau]$, if $u(t) \in D$ for $t \in [0, \tau]$, $Bu(\cdot) \in C([0, \tau]; X)$ and $u(\cdot)$ satisfies the equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s)Bu(s)\,ds \quad \text{for } t \in [0, \tau].$$

A mild solution on $[0, \infty)$ is said to be *global*.

In order to introduce a class of semigroups of locally Lipschitz operators on D, we employ a *comparison function* $g \in C(\mathbb{R}_+^n; \mathbb{R}^n)$ satisfying the following conditions:

(g-i) For each $i = 1, 2, \ldots, n$, $g_i(0) \geqslant 0$.
(g-ii) For each $i = 1, 2, \ldots, n$, $g_i(r)$ is nondecreasing in r_j with $j \neq i$.
(g-iii) For each $\alpha \in \mathbb{R}_+^n$, the Cauchy problem

$$p'(t) = g\big(p(t)\big) \quad \text{for } t \geqslant 0, \quad \text{and} \quad p(0) = \alpha$$

has a global maximal solution $m(t; \alpha)$ on $[0, \infty)$.

Definition 2.2. A one-parameter family $\{S(t);\ t \geqslant 0\}$ of locally Lipschitz operators from D into itself is called a *semigroup of locally Lipschitz operators on D with respect to the functional* φ, if it satisfies the following conditions:

(S1) $S(0)x = x$ for $x \in D$, and $S(t+s)x = S(t)S(s)x$ for $s, t \geqslant 0$ and $x \in D$.
(S2) For each $x \in D$, $S(\cdot)x : [0, \infty) \to X$ is continuous.
(S3) For each $\tau \geqslant 0$ and $\alpha \in \mathbb{R}_+^n$ there exists $L_{\tau,\alpha} > 0$ such that

$$\big\|S(t)x - S(t)y\big\| \leqslant L_{\tau,\alpha}\|x - y\| \quad \text{for } x, y \in D_\alpha \text{ and } t \in [0, \tau].$$

(S4) $\varphi(S(t)x) \leqslant m(t; \varphi(x))$ for $x \in D$ and $t \geqslant 0$.

The following is the main theorem which is an extension of the main results in [5,12].

Theorem 2.3. *Assume that* (A), (φ), (B), (g) *hold. Then, the following two statements are equivalent*:

(i) *There exists a semigroup* $\{S(t);\ t \geqslant 0\}$ *of locally Lipschitz operators on* D *with respect to* φ *such that*

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds \quad \text{for } x \in D \text{ and } t \geqslant 0.$$

(ii) *The following three conditions are satisfied*:

(ii-1) *There exist* $\tau > 0$ *and a family* $\{V_\alpha(\cdot,\cdot,\cdot);\ \alpha \in \mathbb{R}^n_+\}$ *of nonnegative functionals on* $[0,\tau] \times X \times X$ *such that*

(V1) *for each* $\alpha \in \mathbb{R}^n_+$ *and* $x, y \in D_\alpha$, $V_\alpha(\cdot, x, y) : [0,\tau] \to \mathbb{R}_+$ *is continuous,*

(V2) *for each* $\alpha \in \mathbb{R}^n_+$ *there exists* $L(\alpha) \geqslant 0$ *such that*

$$\left|V_\alpha(t,x,y) - V_\alpha(t,\hat{x},\hat{y})\right| \leqslant L(\alpha)\big(\|x-\hat{x}\| + \|y-\hat{y}\|\big)$$

for $(t,x,y), (t,\hat{x},\hat{y}) \in [0,\tau] \times X \times X$,

(V3) *for each* $\alpha \in \mathbb{R}^n_+$ *there exist* $M(\alpha) \geqslant m(\alpha) > 0$ *such that*

$$m(\alpha)\|x-y\| \leqslant V_\alpha(t,x,y) \leqslant M(\alpha)\|x-y\|$$

for $(t,x,y) \in [0,\tau] \times D_\alpha \times D_\alpha$.

(ii-2) *For each* $\alpha \in \mathbb{R}^n_+$ *there exist* $\beta = \beta(\alpha) \in \mathbb{R}^n_+$ *with* $\beta \geqslant \alpha$ *and* $\omega = \omega(\alpha) \geqslant 0$ *such that*

$$\begin{aligned}&\liminf_{h \downarrow 0}\big(V_\beta\big(t+h, T(h)x + hBx, T(h)y + hBy\big) - V_\beta(t,x,y)\big)/h \\ &\quad \leqslant \omega V_\beta(t,x,y)\end{aligned}$$

for $(t,x,y) \in [0,\tau) \times D_\alpha \times D_\alpha$.

(ii-3) *For each* $x \in D$ *and* $\varepsilon > 0$ *there exist* $\delta \in (0,\varepsilon]$ *and* $x_\delta \in D$ *such that*

$$\left\|T(\delta)x + \delta Bx - x_\delta\right\| \leqslant \varepsilon\delta \quad \text{and} \quad \big(\varphi(x_\delta) - \varphi(x)\big)/\delta \leqslant g^\varepsilon\big(\varphi(x)\big),$$

where $g^\varepsilon = (g_i^\varepsilon)_{i=1}^n$ *and* $g_i^\varepsilon(p) = g_i(p) + \varepsilon$ *for* $i = 1, 2, \ldots, n$.

Proof of Theorem 2.3 (i) ⇒ (ii). Let $\tau > 0$ and consider the functional

$$V(t,x,y) := \sup\big\{\left\|S(\sigma)x - S(\sigma)y\right\|;\ 0 \leqslant \sigma \leqslant \tau - t\big\}$$

for $(t,x,y) \in [0,\tau] \times D \times D$. Then, we see by condition (S2) that the definition of V makes sense and

(a) for each $x, y \in D$, $V(\cdot,x,y) : [0,\tau] \to X$ is continuous.

The following conditions (b) and (c) are deduced from condition (S1):

(b) For each $(t,x,y) \in [0,\tau] \times D \times D$, $\|x-y\| \leqslant V(t,x,y)$.

(c) For each $(t,x,y) \in [0,\tau) \times D \times D$ and $h \geqslant 0$ with $t+h \leqslant \tau$,

$$V\big(t+h, S(h)x, S(h)y\big) \leqslant V(t,x,y). \tag{2.1}$$

Set $M(\alpha) = \sup\{V(t,\xi,\eta)/\|\xi-\eta\|;\ (t,\xi,\eta) \in [0,\tau] \times D_\alpha \times D_\alpha,\ \xi \neq \eta\}$ for each $\alpha \in \mathbb{R}^n_+$. Then for each $\alpha \in \mathbb{R}^n_+$, we have $M(\alpha) < \infty$ by condition (S3), and

$$V(t,\xi,\eta) \leqslant M(\alpha)\|\xi-\eta\| \quad \text{for } (t,\xi,\eta) \in [0,\tau] \times D_\alpha \times D_\alpha. \tag{2.2}$$

Let $\alpha \in \mathbb{R}^n_+$. Then, since $V(t,\xi,\eta) - M(\alpha)(\|x-\xi\| + \|y-\eta\|) \leqslant M(\alpha)\|x-y\|$ for $(t,x,y) \in [0,\tau] \times X \times X$ and $\xi, \eta \in D_\alpha$ (by (2.2)), a functional V_α on $[0,\tau] \times X \times X$ can be defined by

$$V_\alpha(t,x,y) = \sup\bigl\{V(t,\xi,\eta) - M(\alpha)\bigl(\|x-\xi\| + \|y-\eta\|\bigr);\ \xi,\eta \in D_\alpha\bigr\} \vee 0.$$

We begin by showing that the family $\{V_\alpha(\cdot,\cdot,\cdot);\ \alpha \in \mathbb{R}^n_+\}$ satisfies condition (ii-1). Since $V(t,\cdot,\cdot)$ is a metric on D and satisfies (2.2), we have

$$\begin{aligned} V(t,\xi,\eta) &\leqslant V(t,x,y) + V(t,\xi,x) + V(t,y,\eta) \\ &\leqslant V(t,x,y) + M(\alpha)\bigl(\|x-\xi\| + \|y-\eta\|\bigr) \end{aligned}$$

for $(t,x,y) \in [0,\tau] \times D_\alpha \times D_\alpha$ and $\xi, \eta \in D_\alpha$. This implies that $V_\alpha(t,x,y) \leqslant V(t,x,y)$ for $(t,x,y) \in [0,\tau] \times D_\alpha \times D_\alpha$. Since the converse inequality follows from the definition of V_α with $(\xi,\eta) = (x,y) \in D_\alpha \times D_\alpha$, we have

$$V_\alpha(t,x,y) = V(t,x,y) \quad \text{for } (t,x,y) \in [0,\tau] \times D_\alpha \times D_\alpha. \tag{2.3}$$

It follows from (a), (b) and (2.2) that (V1) and (V3) are satisfied. Since

$$\begin{aligned} &V(t,\xi,\eta) - M(\alpha)\bigl(\|x-\xi\| + \|y-\eta\|\bigr) - \bigl(V(t,\xi,\eta) - M(\alpha)\bigl(\|\hat{x}-\xi\| + \|\hat{y}-\eta\|\bigr)\bigr) \\ &\quad \leqslant M(\alpha)\bigl(\|x-\hat{x}\| + \|y-\hat{y}\|\bigr) \end{aligned}$$

for $(t,x,y), (t,\hat{x},\hat{y}) \in [0,\tau] \times X \times X$ and $\xi, \eta \in D_\alpha$, condition (V2) is satisfied. Condition (ii-1) is thus shown to be satisfied.

To check condition (ii-2), let $\alpha \in \mathbb{R}^n_+$ and $(t,x,y) \in [0,\tau) \times D_\alpha \times D_\alpha$, and set $\beta = (\beta_i)_{i=1}^n$ where $\beta_i = \alpha_i + 1$ for $i = 1,2,\ldots,n$. Then, there exists $h_0 > 0$ such that $t + h_0 \leqslant \tau$ and $m(h;\alpha) \leqslant \beta$ for $h \in [0,h_0]$. Since $\varphi(S(h)x) \leqslant m(h;\varphi(x)) \leqslant \beta$ for $h \in [0,h_0]$ (by condition (S4)), the inequality (2.1) combined with (2.3) implies that $V_\beta(t+h, S(h)x, S(h)y) \leqslant V_\beta(t,x,y)$ for $h \in [0,h_0]$. It follows that

$$\begin{aligned} &\bigl(V_\beta\bigl(t+h, T(h)x + hBx, T(h)y + hBy\bigr) - V_\beta(t,x,y)\bigr)/h \\ &\quad \leqslant \bigl(V_\beta\bigl(t+h, T(h)x + hBx, T(h)y + hBy\bigr) - V_\beta\bigl(t+h, S(h)x, S(h)y\bigr)\bigr)/h \\ &\quad \leqslant L(\beta)\bigl(\bigl\|T(h)x + hBx - S(h)x\bigr\| + \bigl\|T(h)y + hBy - S(h)y\bigr\|\bigr)/h, \end{aligned}$$

and the right-hand side vanishes as $h \downarrow 0$ because $S(\cdot)x$ is a mild solution to (SP; x) and $\lim_{h\downarrow 0} h^{-1}\int_0^h T(h-s)BS(s)x\,ds = Bx$. This shows that condition (ii-2) is satisfied with $\omega = 0$. Let $x \in D$. Since $S(h)x \in D$ for all $h > 0$, $\lim_{h\downarrow 0}(T(h)x + hBx - S(h)x)/h = 0$ and $\limsup_{h\downarrow 0}(\varphi(S(h)x) - \varphi(x))/h \leqslant \limsup_{h\downarrow 0}(m(h;\varphi(x)) - m(0;\varphi(x)))/h = g(\varphi(x))$, condition (ii-3) is satisfied. □

Remark 2.4. As is shown in the proof of the implication (i) ⇒ (ii), the constructed family $\{V_\alpha(\cdot,\cdot,\cdot);\ \alpha \in \mathbb{R}^n_+\}$ of nonnegative functionals on $[0,\tau] \times X \times X$ has an additional property

$$V_\alpha(t,x,y) = V(t,x,y) \quad \text{for } (t,x,y) \in [0,\tau] \times D_\alpha \times D_\alpha,$$

where $V(t,\cdot,\cdot)$ is a metric on D satisfying the following three conditions:

(v1) For each $x, y \in D$, $V(\cdot,x,y) : [0,\tau] \to X$ is continuous.
(v2) For each $(t,x,y) \in [0,\tau] \times D \times D$, $\|x-y\| \leqslant V(t,x,y)$.
(v3) For each $\alpha \in \mathbb{R}^n_+$ there exists $M(\alpha) > 0$ such that

$$V(t,x,y) \leqslant M(\alpha)\|x-y\| \quad \text{for } (t,x,y) \in [0,\tau] \times D_\alpha \times D_\alpha.$$

A family $\{V_\alpha(\cdot,\cdot,\cdot);\ \alpha \in \mathbb{R}^n_+\}$ of nonnegative functionals on $[0,\tau]\times X\times X$ just satisfying conditions (ii-1) and (ii-2) is considered so that the result can be applied to concrete problems as easily as possible, although the above-mentioned property is necessary for the existence of a semigroup of locally Lipschitz operators associated with semilinear evolution equations.

The following asserts that conditions (ii-1) and (ii-2) together ensure the uniqueness of mild solutions to the Cauchy problem for (SP).

Proposition 2.5. *Let $\bar\tau > 0$, $\alpha \in \mathbb{R}^n_+$ and $x, \hat x \in D$. Let $u, \hat u : [0,\bar\tau] \to X$ be mild solutions to* (SP; x) *and* (SP; $\hat x$) *on $[0,\bar\tau]$, respectively, satisfying $u(t), \hat u(t) \in D_\alpha$ for $t \in [0,\bar\tau]$. Suppose that* (ii-1) *and* (ii-2) *in Theorem* 2.3 *are satisfied. Then there exist $\overline{M}_\alpha > 0$ and $\bar\omega_\alpha > 0$ such that*

$$\|u(t) - \hat u(t)\| \leqslant \overline{M}_\alpha \exp(\bar\omega_\alpha t)\|x - \hat x\| \quad \textit{for } t \in [0,\bar\tau].$$

Proof. Let β and ω be a vector in $\mathbb{R}^n_+$ and a nonnegative number in condition (ii-2) respectively depending only on the given vector $\alpha \in \mathbb{R}^n_+$. Let $\sigma \in [0,\tau]$, where $\tau > 0$ is a number satisfying condition (ii-1). Let l be a nonnegative integer such that $\sigma + l\tau \leqslant \bar\tau$. Then, we observe by (V2) that the mapping $t \mapsto V_\beta(t, u(t+l\tau), \hat u(t+l\tau))$ is continuous on $[0,\sigma)$. Since

$$u(t+h+l\tau) = T(h)u(t+l\tau) + \int_{t+l\tau}^{t+l\tau+h} T(t+h+l\tau-s)Bu(s)\,ds$$

for sufficiently small $h > 0$ (which follows from the definition of mild solutions to (SP; x) and the semigroup property of $\{T(t);\ t \geqslant 0\}$) and since

$$\lim_{h\downarrow 0} h^{-1} \int_{t+l\tau}^{t+l\tau+h} \|T(t+h+l\tau-s)Bu(s) - Bu(t+l\tau)\|\,ds = 0,$$

we find, by (ii-2),

$$D_+V_\beta\big(t, u(t+l\tau), \hat u(t+l\tau)\big) \leqslant \omega V_\beta\big(t, u(t+l\tau), \hat u(t+l\tau)\big)$$

for $t \in [0,\sigma)$, where D_+ denotes the lower right Dini derivative. It follows that

$$V_\beta\big(t, u(t+l\tau), \hat u(t+l\tau)\big) \leqslant e^{\omega t} V_\beta\big(0, u(l\tau), \hat u(l\tau)\big) \tag{2.4}$$

for $t \in [0,\sigma]$, $l \geqslant 0$ with $\sigma + l\tau \leqslant \bar\tau$ and $\sigma \in [0,\tau]$.

Now, let $t \in [0,\bar\tau]$. By $[t/\tau]$ we denote the integer part of t/τ. Since $t = [t/\tau]\tau + \sigma$ for some $\sigma \in [0,\tau)$, an application of (2.4) and (V3) gives

$$\|u(t) - \hat u(t)\| \leqslant \big(M(\beta)/m(\beta)\big)^{[t/\tau]+1} \exp\big(\omega\big(\sigma + [t/\tau]\tau\big)\big)\|u(0) - \hat u(0)\|.$$

The desired result is thus obtained by setting $\overline{M}_\alpha = M(\beta)/m(\beta)$ and $\bar\omega_\alpha = \omega + \tau^{-1}\log(M(\beta)/m(\beta))$. □

Proposition 2.6. *Suppose that* (ii-1) *and* (ii-2) *in Theorem* 2.3 *are satisfied. Suppose that for each $x \in D$ there exist $\bar\tau > 0$ and a mild solution u to* (SP; x) *on $[0,\bar\tau]$ satisfying $\varphi(u(t)) \leqslant m(t;\varphi(x))$ for $t \in [0,\bar\tau]$. Then for every $x \in D$ there exists a global mild solution u to* (SP; x) *satisfying*

$$\varphi\big(u(t)\big) \leqslant m\big(t;\varphi(x)\big) \quad \textit{for } t \geqslant 0.$$

Proof. Let $x \in D$. By $\tau_{\max}$ we denote the supremum of positive numbers $\bar{\tau}$ such that the (SP; x) admits a mild solution u on $[0, \bar{\tau}]$ satisfying $\varphi(u(t)) \leqslant m(t; \varphi(x))$ for $t \in [0, \bar{\tau}]$. Then, by assumption we see that $\tau_{\max} > 0$. By the uniqueness of mild solutions (Proposition 2.5), there exists $u \in C([0, \tau_{\max}); X)$ such that $u(t) = T(t)x + \int_0^t T(t-s)Bu(s)\,ds$ and $\varphi(u(t)) \leqslant m(t; \varphi(x))$ for $t \in [0, \tau_{\max})$. Once the fact $\tau_{\max} = \infty$ is proved, the proposition is true. Now, assume to the contrary that $\tau_{\max} < \infty$, and set $\alpha = (\alpha_i)_{i=1}^n$, where $\alpha_i = \sup\{m_i(t; \varphi(x));\ t \in [0, \tau_{\max}]\}$. Then, by Proposition 2.5 we have

$$\left\| u(t+h) - u(t) \right\| \leqslant \overline{M}_\alpha \exp(\bar{\omega}_\alpha t) \left\| u(h) - u(0) \right\|$$

for $t, t+h \in [0, \tau_{\max})$. Since X is a complete metric space and D_α is closed in X, this implies that $u(t)$ is convergent to some $y \in D_\alpha$ as $t \uparrow \tau_{\max}$. By assumption, there exist $\delta > 0$ and a mild solution w to (SP; y) on $[0, \delta]$ satisfying $\varphi(w(t)) \leqslant m(t; \varphi(y))$ for $t \in [0, \delta]$. The function $\bar{u} : [0, \tau_{\max} + \delta] \to X$, defined by $\bar{u}(t) = u(t)$ for $t \in [0, \tau_{\max})$, and $\bar{u}(t) = w(t - \tau_{\max})$ for $t \in [\tau_{\max}, \tau_{\max} + \delta]$, is a mild solution to (SP; x) on $[0, \tau_{\max} + \delta]$ satisfying the growth condition $\varphi(\bar{u}(t)) \leqslant m(t; \varphi(x))$ for $t \in [0, \tau_{\max} + \delta]$. This contradicts the definition of $\tau_{\max}$. $\square$

By [17, Proposition 2.5] and the Feller renorming technique [4], we may assume, without loss of generality, that the (C_0) semigroup $\{T(t);\ t \geqslant 0\}$ on X is contractive, in showing the implication (ii) $\Rightarrow$ (i) of Theorem 2.3. The proof is divided into two parts. One is the construction of approximate solutions and the other is the convergence of a sequence of approximate solutions to a mild solution to the Cauchy problem for (SP) which forms a semigroups of locally Lipschitz operators. They will be discussed in Sections 3 and 4, respectively.

3. Construction of approximate solutions

To discuss the construction of approximate solutions, we need the local uniformity of condition (ii-3) (Proposition 3.4) which is proved by a sequence of lemmas.

Lemma 3.1. *Let $\alpha \in \mathbb{R}_+^n$ and $v_0 \in D_\alpha$. Let $\bar{h}$, $\bar{r}$, $\overline{M}$ and $\bar{\varepsilon}$ be positive numbers such that*

$$\|Bx\| \leqslant \overline{M} \quad \text{for } x \in U[v_0, \bar{r}] \cap D_\alpha,$$
$$\bar{h}(\overline{M} + \bar{\varepsilon}) + \sup_{s \in [0, \bar{h}]} \left\| T(s)v_0 - v_0 \right\| \leqslant \bar{r},$$

where $U[v_0, \bar{r}]$ denotes the closed ball with center v_0 and radius $\bar{r}$. Let $\delta \in [0, \bar{h}]$, $w_0 \in D_\alpha$ and $\sigma > 0$ be such that

$$\sigma + \delta \leqslant \bar{h} \quad \text{and} \quad \left\| w_0 - T(\delta)v_0 \right\| \leqslant \delta(\overline{M} + \bar{\varepsilon}).$$

Assume that there exists a sequence $\{(s_i, w_i)\}_{i=0}^N$ in $[0, \sigma] \times D_\alpha$ such that

$$0 = s_0 < s_1 < \cdots < s_N \leqslant \sigma, \tag{3.1}$$
$$\left\| T(s_i - s_{i-1})w_{i-1} + (s_i - s_{i-1})Bw_{i-1} - w_i \right\| \leqslant \bar{\varepsilon}(s_i - s_{i-1}) \quad \text{for } i = 1, 2, \ldots, N. \tag{3.2}$$

Then the following assertions hold:

(i) $\|T(s_j - s_k)w_k - w_j\| \leqslant (s_j - s_k)(\overline{M} + \bar{\varepsilon})$ *for* $0 \leqslant k \leqslant j \leqslant N$.
(ii) $\|w_j - T(s_j + \delta)v_0\| \leqslant (s_j + \delta)(\overline{M} + \bar{\varepsilon})$ *for* $0 \leqslant j \leqslant N$.
(iii) $w_j \in U[v_0, \bar{r}]$ *and* $\|Bw_j\| \leqslant \overline{M}$ *for* $0 \leqslant j \leqslant N$.

Proof. We prove the lemma by induction. In the case of $j=0$, assertion (i) is obvious and assertion (ii) follows by assumption. Since $\delta \leqslant \bar{h}$ we have

$$\|w_0 - v_0\| \leqslant \|w_0 - T(\delta)v_0\| + \|T(\delta)v_0 - v_0\| \leqslant \delta(\overline{M} + \bar{\varepsilon}) + \|T(\delta)v_0 - v_0\| \leqslant \bar{r},$$

so that assertion (iii) holds for $j=0$. Now, let $1 \leqslant i \leqslant N$ and suppose that assertions (i) through (iii) hold for $j = i-1$. To prove assertion (i) with $j=i$ we have only to consider the case where $0 \leqslant k \leqslant i-1$. Since

$$\begin{aligned} T(s_i - s_k)w_k - w_i = {} & T(s_i - s_{i-1})\bigl(T(s_{i-1} - s_k)w_k - w_{i-1}\bigr) \\ & + \bigl(T(s_i - s_{i-1})w_{i-1} + (s_i - s_{i-1})Bw_{i-1} - w_i\bigr) \\ & - (s_i - s_{i-1})Bw_{i-1}, \end{aligned}$$

it follows from the hypotheses (i) and (iii) with $j = i-1$, and condition (3.2) that assertion (i) is valid for $j=i$. Since

$$w_i - T(s_i + \delta)v_0 = w_i - T(s_i - s_{i-1})w_{i-1} + T(s_i - s_{i-1})\bigl(w_{i-1} - T(s_{i-1} + \delta)v_0\bigr),$$

we deduce from (i) with $(j,k) = (i, i-1)$ and (ii) with $j = i-1$ that assertion (ii) holds for $j=i$. Since $\|w_i - v_0\| \leqslant (s_i + \delta)(\overline{M} + \bar{\varepsilon}) + \|T(s_i + \delta)v_0 - v_0\|$ (by (ii) with $j=i$) and since $s_i + \delta \leqslant \bar{h}$, assertion (iii) is true for $j=i$. $\square$

Lemma 3.2. *Let $\alpha \in \mathbb{R}^n_+$ and $v_0 \in D_\alpha$. Let $\bar{h}$, $\bar{r}$, $\overline{M}$, $\bar{\varepsilon}$ and $\bar{\eta}$ be positive numbers such that*

$$\|Bx\| \leqslant \overline{M} \quad \text{and} \quad \|Bx - Bv_0\| \leqslant \bar{\eta} \quad \text{for } x \in U[v_0, \bar{r}] \cap D_\alpha, \tag{3.3}$$

$$\sup_{s \in [0,\bar{h}]} \|T(s)Bv_0 - Bv_0\| \leqslant \bar{\eta}, \tag{3.4}$$

$$\bar{h}(\overline{M} + \bar{\varepsilon}) + \sup_{s \in [0,\bar{h}]} \|T(s)v_0 - v_0\| \leqslant \bar{r}. \tag{3.5}$$

Let $\delta \in [0, \bar{h}]$, $w_0 \in D_\alpha$ and $\sigma > 0$ be such that

$$\sigma + \delta \leqslant \bar{h} \quad \text{and} \quad \|w_0 - T(\delta)v_0\| \leqslant \delta(\overline{M} + \bar{\varepsilon}).$$

Then the following assertions are true:

(i) *If a sequence $\{(s_i, w_i)\}_{i=0}^N$ in $[0,\sigma] \times D_\alpha$ satisfies* (3.1) *and* (3.2)*, then*

$$\|T(s_N)w_0 + s_N Bw_0 - w_N\| \leqslant s_N(\bar{\varepsilon} + 3\bar{\eta}). \tag{3.6}$$

(ii) *If a sequence $\{(s_i, w_i)\}_{i=0}^\infty$ in $[0,\sigma) \times D_\alpha$ satisfies*

$$0 = s_0 < s_1 < \cdots < s_i < \cdots < \sigma \quad \text{and} \quad \lim_{i\to\infty} s_i = \sigma, \tag{3.7}$$

$$\|T(s_i - s_{i-1})w_{i-1} + (s_i - s_{i-1})Bw_{i-1} - w_i\| \leqslant \bar{\varepsilon}(s_i - s_{i-1}) \quad \text{for } i = 1, 2, \ldots \tag{3.8}$$

then there exists $\bar{w} \in D_\alpha$ such that $\bar{w} = \lim_{i\to\infty} w_i$ and

$$\|T(\sigma)w_0 + \sigma Bw_0 - \bar{w}\| \leqslant \sigma(\bar{\varepsilon} + 3\bar{\eta}).$$

Proof. To prove assertion (i), we use (3.2), the inequality (3.3) together with (iii) of Lemma 3.1, and (3.4) to estimate

$$\begin{aligned}&T(s_N - s_i)\big(T(s_i - s_{i-1})w_{i-1} + (s_i - s_{i-1})Bw_{i-1} - w_i\big)\\&\quad + (s_i - s_{i-1})T(s_N - s_i)(Bv_0 - Bw_{i-1})\\&\quad + (s_i - s_{i-1})\big(Bv_0 - T(s_N - s_i)Bv_0\big) + (s_i - s_{i-1})(Bw_0 - Bv_0)\end{aligned}$$

for $1 \leqslant i \leqslant N$. This yields that

$$\big\|T(s_N - s_{i-1})w_{i-1} + (s_i - s_{i-1})Bw_0 - T(s_N - s_i)w_i\big\| \leqslant (s_i - s_{i-1})(\bar\varepsilon + 3\bar\eta)$$

for $1 \leqslant i \leqslant N$. Adding the inequality above from $i = 1$ up to N, we obtain the desired inequality (3.6). To prove (ii), let $\{(s_i, w_i)\}_{i=1}^{\infty}$ be a sequence in $[0, \sigma) \times D_\alpha$ satisfying (3.7) and (3.8). Since

$$\begin{aligned}w_i - w_j &= \big(w_i - T(s_i - s_k)w_k\big) + \big(T(s_i - s_k)w_k - T(s_j - s_k)w_k\big)\\&\quad + \big(T(s_j - s_k)w_k - w_j\big)\end{aligned}$$

for $i, j \geqslant k \geqslant 0$, we deduce from (i) of Lemma 3.1 and the strong continuity of $T(\cdot)$ in X on $[0, \infty)$ that $\limsup_{i,j\to\infty} \|w_i - w_j\| \leqslant 2(\sigma - s_k)(\overline{M} + \bar\varepsilon)$ for all $k \geqslant 0$. Since $\lim_{k\to\infty} s_k = \sigma$ and D_α is closed in X, we find $\bar w \in D_\alpha$ such that $\bar w = \lim_{i\to\infty} w_i$. The desired result is obtained by letting $N \to \infty$ in (3.6). $\square$

To specify the growth of approximate solutions, we use the nonextensible maximal solution $m^\varepsilon(t; \alpha)$ to the Cauchy problem for the finite system

$$q'(t) = g^\varepsilon\big(q(t)\big) \quad \text{for } t \geqslant 0,$$

with initial condition $q(0) = \alpha$, where $\varepsilon > 0$ and $\alpha \in \mathbb{R}^n_+$. Let $\tau^\varepsilon(\alpha)$ denote the maximal existence time of the maximal solution $m^\varepsilon(t; \alpha)$, for each $\alpha \in \mathbb{R}^n_+$. Then it is known [13, Section 1.5] that the following assertions hold:

(m1) If $\beta \geqslant \alpha \geqslant 0$ then $\tau^\varepsilon(\beta) \leqslant \tau^\varepsilon(\alpha)$ and $m^\varepsilon(t; \alpha) \leqslant m^\varepsilon(t; \beta)$ for $t \in [0, \tau^\varepsilon(\beta))$.
(m2) If $s \in [0, \tau^\varepsilon(\alpha))$ then $\tau^\varepsilon(m^\varepsilon(s; \alpha)) = \tau^\varepsilon(\alpha) - s$ and $m^\varepsilon(t + s; \alpha) = m^\varepsilon(t; m^\varepsilon(s; \alpha))$ for $t \in [0, \tau^\varepsilon(\alpha) - s)$.
(m3) $\lim_{\varepsilon\downarrow 0} \tau^\varepsilon(\alpha) = \infty$ and $\lim_{\varepsilon\downarrow 0} m^\varepsilon(t; \alpha) = m(t; \alpha)$ uniformly on every compact subinterval of $[0, \infty)$.

Lemma 3.3. *Suppose that condition* (ii-3) *in Theorem* 2.3 *is satisfied. Then, for each $x \in D$ and $\varepsilon > 0$ there exist $\delta \in (0, \varepsilon]$ and $x_\delta \in D$ such that*

$$\big\|x_\delta - T(\delta)x - \delta Bx\big\| \leqslant \varepsilon\delta \quad \textit{and} \quad \varphi(x_\delta) \leqslant m^\varepsilon\big(\delta; \varphi(x)\big).$$

Proof. Let $x \in D$ and $\varepsilon > 0$. Then the vector-valued function $p = (p_i)_{i=1}^n \in C([0, \infty); \mathbb{R}^n)$, defined by

$$p_i(t) = \varphi_i(x) + t\big(g_i\big(\varphi(x)\big) + \varepsilon/2\big)$$

for $1 \leqslant i \leqslant n$, satisfies $p(0) = \varphi(x) = m^\varepsilon(0; \varphi(x))$ and $p_i'(t) = g_i(\varphi(x)) + \varepsilon/2$ for $t \geqslant 0$ and $1 \leqslant i \leqslant n$. Since $p_i'(0) < g_i(m^\varepsilon(0; \varphi(x))) + \varepsilon = (d/dt)m_i^\varepsilon(0; \varphi(x))$ for $1 \leqslant i \leqslant n$, there exists $\tau \in (0, \tau^\varepsilon(\varphi(x)))$ such that

$$p(t) \leqslant m^\varepsilon\big(t; \varphi(x)\big)$$

for $t \in [0, \tau]$. By condition (ii-3), there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in D and a null sequence $\{\delta_k\}_{k=1}^{\infty}$ of positive numbers such that $\|x_k - T(\delta_k)x - \delta_k Bx\| \leqslant \delta_k/k$ and $(\varphi_i(x_k) - \varphi_i(x))/\delta_k \leqslant g_i(\varphi(x)) + 1/k$ for $1 \leqslant i \leqslant n$. If k is chosen so that $1/k \leqslant \varepsilon/2$ and $\delta_k \leqslant (\tau \wedge \varepsilon)$, then we observe that (x_k, δ_k) is the desired element, since $\varphi_i(x_k) \leqslant \varphi_i(x) + \delta_k(g_i(\varphi(x)) + \varepsilon/2) = p_i(\delta_k) \leqslant m_i^{\varepsilon}(\delta_k; \varphi(x))$. □

The next proposition shows that the subtangential condition (ii-3) holds uniformly in a neighborhood of each element of D.

Proposition 3.4. *Suppose that* (ii-3) *in Theorem* 2.3 *holds. Let $\alpha \in \mathbb{R}_+^n$ and $v_0 \in D_\alpha$. Let $\bar{h}$, $\bar{r}$, $\overline{M}$, $\bar{\eta}$ and $\bar{\varepsilon}$ be positive numbers satisfying*

$$\|Bx\| \leqslant \overline{M} \quad \text{and} \quad \|Bx - Bv_0\| \leqslant \bar{\eta} \quad \text{for } x \in U[v_0, \bar{r}] \cap D_\alpha,$$
$$\sup_{s\in[0,\bar{h}]} \left\|T(s)Bv_0 - Bv_0\right\| \leqslant \bar{\eta},$$
$$\bar{h} < \tau^{\bar{\varepsilon}}\big(\varphi(v_0)\big) \quad \text{and} \quad m^{\bar{\varepsilon}}\big(s; \varphi(v_0)\big) \leqslant \alpha \quad \text{for } s \in [0, \bar{h}],$$
$$\bar{h}(\overline{M} + \bar{\varepsilon}) + \sup_{s\in[0,\bar{h}]} \left\|T(s)v_0 - v_0\right\| \leqslant \bar{r}.$$

Let $\delta \in [0, \bar{h}]$, $w_0 \in D_\alpha$ and assume that

$$\left\|w_0 - T(\delta)v_0\right\| \leqslant \delta(\overline{M} + \bar{\varepsilon}) \quad \text{and} \quad \varphi(w_0) \leqslant m^{\bar{\varepsilon}}\big(\delta; \varphi(v_0)\big).$$

Then, for each $\sigma > 0$ with $\sigma + \delta \leqslant \bar{h}$ there exists $z_0 \in D_\alpha$ such that

$$\left\|T(\sigma)w_0 + \sigma Bw_0 - z_0\right\| \leqslant \sigma(\bar{\varepsilon} + 3\bar{\eta}) \quad \text{and} \quad \varphi(z_0) \leqslant m^{\bar{\varepsilon}}\big(\sigma; \varphi(w_0)\big).$$

Proof. Let $\sigma > 0$ satisfy $\sigma + \delta \leqslant \bar{h}$. We begin by constructing a sequence $\{(s_i, w_i)\}_{i=0}^{\infty}$ in $[0, \sigma) \times D_\alpha$ satisfying the following conditions:

(a) $0 = s_0 < s_1 < \cdots < s_i < \cdots < \sigma$.
(b) $\|T(s_i - s_{i-1})w_{i-1} + (s_i - s_{i-1})Bw_{i-1} - w_i\| \leqslant \bar{\varepsilon}(s_i - s_{i-1})$ for $i = 1, 2, \ldots$.
(c) $\varphi(w_i) \leqslant m^{\bar{\varepsilon}}(s_i - s_{i-1}; \varphi(w_{i-1}))$ for $i = 1, 2, \ldots$.
(d) $\lim_{i\to\infty} s_i = \sigma$.

To this end, assume that a sequence $\{(s_i, w_i)\}_{i=0}^{k-1}$ in $[0, \sigma) \times D_\alpha$ satisfying (a) through (c) can be chosen for some $k \geqslant 1$. Then we define $\bar{h}_k$ by the supremum of numbers $h \in [0, \sigma - s_{k-1})$ such that there exists $x_h \in D$ satisfying

$$\left\|x_h - T(h)w_{k-1} - hBw_{k-1}\right\| \leqslant \bar{\varepsilon}h \quad \text{and} \quad \varphi(x_h) \leqslant m^{\bar{\varepsilon}}\big(h; \varphi(w_{k-1})\big).$$

Since $\bar{h}_k > 0$ by Lemma 3.3, there exist $h_k > 0$ and $w_k \in D$ such that $\bar{h}_k/2 < h_k < \sigma - s_{k-1}$ and such that if we define $s_k = s_{k-1} + h_k$, then the pair (s_k, w_k) satisfies (a) through (c) with $i = k$. Since $s_k + \delta \leqslant \bar{h}$ we apply (m1) and (m2) to condition (c) with $1 \leqslant i \leqslant k$, so that

$$\varphi(w_k) \leqslant m^{\bar{\varepsilon}}\big(s_k; \varphi(w_0)\big) \leqslant m^{\bar{\varepsilon}}\big(s_k + \delta; \varphi(v_0)\big); \tag{3.9}$$

hence $w_k \in D_\alpha$ by assumption. Thus, we can inductively construct a sequence $\{(s_i, w_i)\}_{i=0}^{\infty}$ in $[0, \sigma) \times D_\alpha$ satisfying (a) through (c). To show that condition (d) is satisfied, assume to the contrary that $\bar{s} := \lim_{i\to\infty} s_i < \sigma$. Lemma 3.2(ii) then asserts that the sequence $\{w_i\}_{i=0}^{\infty}$ in D_α is convergent in X to some $\bar{w} \in D_\alpha$. Applying Lemma 3.3, we find $h \in (0, \sigma - \bar{s})$ and $x_h \in D$ such that

$$\|x_h - T(h)\bar{w} - hB\bar{w}\| \leqslant h(\bar{\varepsilon}/2) \quad \text{and} \quad \varphi(x_h) \leqslant m^{\bar{\varepsilon}/2}\big(h; \varphi(\bar{w})\big).$$

Set $\gamma_i = \bar{s} + h - s_{i-1}$ for $i \geqslant 1$. Then, there exits an integer $i_0 \geqslant 1$ such that $i \geqslant i_0$ implies that

$$\bar{h}_i < \gamma_i < \sigma - s_{i-1}, \tag{3.10}$$

$$\|x_h - T(\gamma_i)w_{i-1} - \gamma_i B w_{i-1}\| \leqslant \gamma_i \bar{\varepsilon}, \tag{3.11}$$

since $\bar{h}_i < 2h_i \to 0$, $w_i \to \bar{w}$ and $\gamma_i \to h$ as $i \to \infty$. By an application of (m1) and (m2) we deduce from condition (c) that $\varphi(w_i) \leqslant m^{\bar{\varepsilon}}(s_i - s_j; \varphi(w_j))$ for $0 \leqslant j \leqslant i$. By condition ($\varphi$), a passage to the limit in the above inequality yields that $\varphi(\bar{w}) \leqslant m^{\bar{\varepsilon}}(\bar{s} - s_j; \varphi(w_j))$ for $j \geqslant 0$; hence

$$\varphi(x_h) \leqslant m^{\bar{\varepsilon}}\big(h; \varphi(\bar{w})\big) \leqslant m^{\bar{\varepsilon}}\big(\gamma_i; \varphi(w_{i-1})\big) \tag{3.12}$$

for $i \geqslant i_0$. These facts (3.10) through (3.12) contradict the definition of $\bar{h}_i$. It is thus concluded that (d) is satisfied. Applying Lemma 3.2 to the sequence $\{(s_i, w_i)\}_{i=0}^{\infty}$ constructed above and letting $k \to \infty$ in (3.9), we see that the limit $z_0 := \lim_{i\to\infty} w_i$ is the desired one. □

The following proposition establishes the existence of approximate solutions to the Cauchy problem for (SP).

Proposition 3.5. *Suppose that condition* (ii-3) *in Theorem* 2.3 *is satisfied. Let* $\alpha \in \mathbb{R}^n_+$ *and* $x_0 \in D_\alpha$. *Let* $\bar{\tau} > 0$, $R > 0$, $M_B > 0$, $\varepsilon \in (0, 1]$ *and assume that*

$$\|Bx\| \leqslant M_B \quad \text{for } x \in U[x_0, R] \cap D_\alpha,$$

$$\bar{\tau}(M_B + 1) + \sup_{s\in[0,\bar{\tau}]} \|T(s)x_0 - x_0\| \leqslant R,$$

$$\bar{\tau} < \tau^{\varepsilon}\big(\varphi(x_0)\big) \quad \text{and} \quad m^{\varepsilon}\big(s; \varphi(x_0)\big) \leqslant \alpha \quad \text{for } s \in [0, \bar{\tau}].$$

Then there exists a sequence $\{(t_j, x_j)\}_{j=0}^{\infty}$ *in* $[0, \bar{\tau}) \times D_\alpha$ *satisfying the following conditions*:

(i) $0 = t_0 < t_1 < \cdots < t_j < \cdots < \bar{\tau}$.
(ii) $t_j - t_{j-1} \leqslant \varepsilon$ *for* $j = 1, 2, \ldots$.
(iii) $\|T(t_j - t_{j-1})x_{j-1} + (t_j - t_{j-1})Bx_{j-1} - x_j\| \leqslant (\varepsilon/2)(t_j - t_{j-1})$ *for* $j = 1, 2, \ldots$.
(iv) *If* $x \in D_\alpha$ *satisfies*

$$\|x - x_{j-1}\| \leqslant (t_j - t_{j-1})(M_B + 1) + \sup_{s\in[0,t_j - t_{j-1}]} \|T(s)x_{j-1} - x_{j-1}\|,$$

then $\|Bx - Bx_{j-1}\| \leqslant \varepsilon/8$ *for* $j = 1, 2, \ldots$.
(v) $(t_j - t_{j-1})(M_B + 1) + \sup_{s\in[0,t_j - t_{j-1}]} \|T(s)x_{j-1} - x_{j-1}\| \leqslant \varepsilon$ *for* $j = 1, 2, \ldots$.
(vi) *If* $s \in [0, t_j - t_{j-1}]$, *then* $\|T(s)Bx_{j-1} - Bx_{j-1}\| \leqslant \varepsilon/8$ *for* $j = 1, 2, \ldots$.
(vii) $\varphi(x_j) \leqslant m^{\varepsilon}(t_j - t_{j-1}; \varphi(x_{j-1}))$ *for* $j = 1, 2, \ldots$.
(viii) $\lim_{j\to\infty} t_j = \bar{\tau}$.

Proof. To construct inductively the desired sequence, let $i \geqslant 1$ and assume that a sequence $\{(t_j, x_j)\}_{j=0}^{i-1}$ in $[0, \bar{\tau}) \times D_\alpha$ can be chosen so that it satisfies (i) through (vii). Then we define $\bar{h}_i$ by the supremum of numbers $h \in [0, \varepsilon]$ such that $h < \bar{\tau} - t_{i-1}$, $r :=$

$h(M_B+1)+\sup_{s\in[0,h]}\|T(s)x_{i-1}-x_{i-1}\| \leqslant \varepsilon$, $\|Bx-Bx_{i-1}\| \leqslant \varepsilon/8$ for $x\in U[x_{i-1},r]\cap D_\alpha$, and $\|T(s)Bx_{i-1}-Bx_{i-1}\| \leqslant \varepsilon/8$ for $s\in[0,h]$. Since $\bar{h}_i>0$ by condition (B) and the strong continuity of $T(\cdot)$ in X on $[0,\infty)$, there exists $h_i\in(0,\varepsilon]$ such that $\bar{h}_i/2<h_i<\bar{\tau}-t_{i-1}$, $r_i\leqslant\varepsilon$,

$$\|Bx-Bx_{i-1}\| \leqslant \varepsilon/8 \quad \text{for } x\in U[x_{i-1},r_i]\cap D_\alpha, \tag{3.13}$$

$$\big\|T(s)Bx_{i-1}-Bx_{i-1}\big\| \leqslant \varepsilon/8 \quad \text{for } s\in[0,h_i], \tag{3.14}$$

where

$$r_i=h_i(M_B+1)+\sup_{s\in[0,h_i]}\big\|T(s)x_{i-1}-x_{i-1}\big\|. \tag{3.15}$$

If we set $t_i=t_{i-1}+h_i$, then conditions (i), (ii), and (iv) through (vi) are satisfied. We shall apply Proposition 3.4 to show the existence of an element $x_i\in D_\alpha$ satisfying conditions (iii) and (vii). By an application of (iii) of Lemma 3.1 with $v_0=w_0=x_0$, $\delta=0$, $\sigma=\bar{h}=\bar{\tau}$, $\overline{M}=M_B$, $\bar{r}=R$ and $\bar{\varepsilon}=\varepsilon/2$ to the sequence $\{(t_j,x_j)\}_{j=0}^{i-1}$, we have $\|Bx_{i-1}\|\leqslant M_B$. This inequality combined with (3.13) implies that

$$\|Bx\| \leqslant M_B+\varepsilon/8 \quad \text{for } x\in U[x_{i-1},r_i]\cap D_\alpha. \tag{3.16}$$

By (3.13) through (3.16), and the inequalities

$$\begin{aligned}
&\tau^{\varepsilon/8}\big(\varphi(x_{i-1})\big) \geqslant \tau^{\varepsilon}\big(m^{\varepsilon}\big(t_{i-1};\varphi(x_0)\big)\big)=\tau^{\varepsilon}\big(\varphi(x_0)\big)-t_{i-1}>t_i-t_{i-1}=h_i,\\
&m^{\varepsilon/8}\big(s;\varphi(x_{i-1})\big) \leqslant m^{\varepsilon}\big(s;m^{\varepsilon}\big(t_{i-1};\varphi(x_0)\big)\big)\\
&\qquad\qquad\qquad\quad = m^{\varepsilon}\big(t_{i-1}+s;\varphi(x_0)\big) \leqslant \alpha \quad \text{for } s\in[0,h_i],
\end{aligned}$$

we apply Proposition 3.4 with $\overline{M}=M_B+\varepsilon/8$, $\bar{\eta}=\varepsilon/8$, $v_0=w_0=x_{i-1}$, $\delta=0$, $\bar{h}=\sigma=h_i$, $\bar{r}=r_i$ and $\bar{\varepsilon}=\varepsilon/8$ to obtain $x_i\in D_\alpha$ satisfying conditions (iii) and (vii) with $j=i$. Thus, we obtain a sequence $\{(t_j,x_j)\}_{j=1}^{\infty}$ in $[0,\bar{\tau})\times D_\alpha$ satisfying conditions (i) through (vii).

It remains to show that condition (viii) is satisfied. To this end, we assume to the contrary that $\bar{t}:=\lim_{j\to\infty}t_j<\bar{\tau}$. Applying (i) of Lemma 3.1 with $v_0=w_0=x_0$, $\delta=0$, $\sigma=\bar{h}=\bar{\tau}$, $\overline{M}=M_B$, $\bar{r}=R$ and $\bar{\varepsilon}=\varepsilon/2$, we have

$$\|x_i-x_j\| \leqslant \big((t_i-t_k)+(t_j-t_k)\big)(M_B+\varepsilon/2)+\big\|T(t_j-t_k)x_k-T(t_i-t_k)x_k\big\|$$

for $i,j\geqslant k\geqslant 0$. This together with the strong continuity of $T(\cdot)$ in X on $[0,\infty)$ implies that $\limsup_{i,j\to\infty}\|x_i-x_j\|\leqslant 2(\bar{t}-t_k)(M_B+\varepsilon/2)$ for all $k\geqslant 0$. Since $\lim_{k\to\infty}t_k=\bar{t}$, the inequality above shows that the sequence $\{x_j\}$ in D_α is convergent to some $\bar{x}\in D_\alpha$. Take $h\in(0,\varepsilon]$ such that $h<\bar{\tau}-\bar{t}$, $\bar{r}:=h(M_B+1)+\sup_{s\in[0,h]}\|T(s)\bar{x}-\bar{x}\|\leqslant\varepsilon/2$, $\|Bx-B\bar{x}\|\leqslant\varepsilon/16$ for $x\in U[\bar{x},2\bar{r}]\cap D_\alpha$ and $\sup_{s\in[0,h]}\|T(s)B\bar{x}-B\bar{x}\|\leqslant\varepsilon/16$. Then, since $\lim_{i\to\infty}x_i=\bar{x}$, $\lim_{i\to\infty}Bx_i=B\bar{x}$ (by condition (B)) and $T(t)$ is contractive for all $t\geqslant 0$, there exists an integer $i_0\geqslant 1$ such that $h\leqslant\bar{h}_i$ for every $i\geqslant i_0$. Here we have used the fact that $\bar{r}_i:=h(M_B+1)+\sup_{s\in[0,h]}\|T(s)x_{i-1}-x_{i-1}\|$ converges to $\bar{r}$ as $i\to\infty$ and $U[x_{i-1},\bar{r}_i]\subset U[\bar{x},2\bar{r}]$ for sufficiently large $i\geqslant 1$. This contradicts the fact $h>0$, since $\bar{h}_i<2h_i=2(t_i-t_{i-1})\to 0$ as $i\to\infty$. $\square$

4. Generation of semigroups of locally Lipschitz operators

In this section we give the proof of the implication (ii) $\Rightarrow$ (i) of Theorem 2.3, namely, the generation of semigroups of locally Lipschitz operators associated with semilinear evolution equations. For this purpose, we first estimate the difference between approximate solutions by an argument similar to that used in [12] and apply the result to the convergence of the sequence of approximate solutions constructed in the previous section to a mild solution to the Cauchy problem for (SP).

Proposition 4.1. *Let $\alpha \in \mathbb{R}^n_+$ and $\bar{v}_0, \hat{v}_0 \in D_\alpha$. Let $\bar{h} > 0$, $\bar{r} > 0$, $\overline{M} > 0$, $\bar{\eta} > 0$, $\bar{\varepsilon} > 0$, $\hat{h} > 0$, $\hat{r} > 0$, $\widehat{M} > 0$, $\hat{\eta} > 0$ and $\hat{\varepsilon} > 0$. Assume that*

$$\begin{aligned} &\|Bx\| \leqslant \overline{M} \quad \text{and} \quad \|Bx - B\bar{v}_0\| \leqslant \bar{\eta}/8 \quad \text{for } x \in U[\bar{v}_0, \bar{r}] \cap D_\alpha, \\ &\|Bx\| \leqslant \widehat{M} \quad \text{and} \quad \|Bx - B\hat{v}_0\| \leqslant \hat{\eta}/8 \quad \text{for } x \in U[\hat{v}_0, \hat{r}] \cap D_\alpha, \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\sup_{s \in [0,\bar{h}]} \|T(s)B\bar{v}_0 - B\bar{v}_0\| \leqslant \bar{\eta}/8, \\ &\sup_{s \in [0,\hat{h}]} \|T(s)B\hat{v}_0 - B\hat{v}_0\| \leqslant \hat{\eta}/8, \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\bar{h} < \tau^{\bar{\varepsilon}}\big(\varphi(\bar{v}_0)\big) \quad \text{and} \quad m^{\bar{\varepsilon}}\big(s; \varphi(\bar{v}_0)\big) \leqslant \alpha \quad \text{for } s \in [0, \bar{h}], \\ &\hat{h} < \tau^{\hat{\varepsilon}}\big(\varphi(\hat{v}_0)\big) \quad \text{and} \quad m^{\hat{\varepsilon}}\big(s; \varphi(\hat{v}_0)\big) \leqslant \alpha \quad \text{for } s \in [0, \hat{h}], \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\bar{h}(\overline{M} + \bar{\varepsilon} + \bar{\eta}) + \sup_{s \in [0,\bar{h}]} \|T(s)\bar{v}_0 - \bar{v}_0\| \leqslant \bar{r}, \\ &\hat{h}(\widehat{M} + \hat{\varepsilon} + \hat{\eta}) + \sup_{s \in [0,\hat{h}]} \|T(s)\hat{v}_0 - \hat{v}_0\| \leqslant \hat{r}. \end{aligned} \tag{4.4}$$

Let $\bar{\delta} \in [0, \bar{h}]$, $\hat{\delta} \in [0, \hat{h}]$, $\bar{w}_0, \hat{w}_0 \in D_\alpha$ and assume that

$$\begin{aligned} &\|\bar{w}_0 - T(\bar{\delta})\bar{v}_0\| \leqslant \bar{\delta}(\overline{M} + \bar{\varepsilon}) \quad \text{and} \quad \varphi(\bar{w}_0) \leqslant m^{\bar{\varepsilon}}\big(\bar{\delta}; \varphi(\bar{v}_0)\big), \\ &\|\hat{w}_0 - T(\hat{\delta})\hat{v}_0\| \leqslant \hat{\delta}(\widehat{M} + \hat{\varepsilon}) \quad \text{and} \quad \varphi(\hat{w}_0) \leqslant m^{\hat{\varepsilon}}\big(\hat{\delta}; \varphi(\hat{v}_0)\big). \end{aligned} \tag{4.5}$$

Let $\tau_0 \in [0, \tau)$. Then, for each $\sigma > 0$ with $\sigma + \bar{\delta} \leqslant \bar{h}$, $\sigma + \hat{\delta} \leqslant \hat{h}$ and $\sigma + \tau_0 \leqslant \tau$ there exist $\bar{z}_0, \hat{z}_0 \in D_\alpha$ such that

$$\begin{aligned} &\|T(\sigma)\bar{w}_0 + \sigma B\bar{w}_0 - \bar{z}_0\| \leqslant \sigma(\bar{\varepsilon} + \bar{\eta}) \quad \text{and} \quad \varphi(\bar{z}_0) \leqslant m^{\bar{\varepsilon}}\big(\sigma; \varphi(\bar{w}_0)\big), \\ &\|T(\sigma)\hat{w}_0 + \sigma B\hat{w}_0 - \hat{z}_0\| \leqslant \sigma(\hat{\varepsilon} + \hat{\eta}) \quad \text{and} \quad \varphi(\hat{z}_0) \leqslant m^{\hat{\varepsilon}}\big(\sigma; \varphi(\hat{w}_0)\big), \\ &V_\beta(\tau_0 + \sigma, \bar{z}_0, \hat{z}_0) \leqslant \exp(\omega\sigma)\big(V_\beta(\tau_0, \bar{w}_0, \hat{w}_0) + \sigma L(\beta)(\bar{\eta} + \hat{\eta} + \bar{\varepsilon} + \hat{\varepsilon})\big). \end{aligned}$$

Here β and ω are constants appearing in condition (ii-2) *of Theorem* 2.3.

Proof. We begin by constructing a sequence $\{(s_j, \bar{w}_j, \hat{w}_j)\}_{j=0}^\infty$ in $[0, \sigma) \times D_\alpha \times D_\alpha$ satisfying $0 = s_0 < s_1 < \cdots < s_j < \cdots < \sigma$, $\lim_{j\to\infty} s_j = \sigma$ and the following conditions:

(i) For each $j = 1, 2, \ldots,$

$$\begin{aligned} &\|T(s_j - s_{j-1})\bar{w}_{j-1} + (s_j - s_{j-1})B\bar{w}_{j-1} - \bar{w}_j\| \leqslant (\bar{\eta}/2 + \bar{\varepsilon})(s_j - s_{j-1}) \quad \text{and} \\ &\|T(s_j - s_{j-1})\hat{w}_{j-1} + (s_j - s_{j-1})B\hat{w}_{j-1} - \hat{w}_j\| \leqslant (\hat{\eta}/2 + \hat{\varepsilon})(s_j - s_{j-1}). \end{aligned}$$

(ii) For each $j = 1, 2, \ldots,$

$$\varphi(\bar{w}_j) \leqslant m^{\bar{\varepsilon}}\big(s_j - s_{j-1}; \varphi(\bar{w}_{j-1})\big) \quad \text{and} \quad \varphi(\hat{w}_j) \leqslant m^{\hat{\varepsilon}}\big(s_j - s_{j-1}; \varphi(\hat{w}_{j-1})\big).$$

(iii) For each $j = 1, 2, \dots,$

$$\begin{aligned}&\big(V_\beta(s_j+\tau_0,\bar{w}_j,\hat{w}_j) - V_\beta(s_{j-1}+\tau_0,\bar{w}_{j-1},\hat{w}_{j-1})\big)/(s_j - s_{j-1})\\&\quad\leqslant \omega V_\beta(s_{j-1}+\tau_0,\bar{w}_{j-1},\hat{w}_{j-1}) + L(\beta)(\bar{\eta}+\hat{\eta}+\bar{\varepsilon}+\hat{\varepsilon}).\end{aligned}$$

To do this, let $i \geqslant 1$ and suppose that we may choose a sequence $\{(s_j,\bar{w}_j,\hat{w}_j)\}_{j=0}^{i-1}$ in $[0,\sigma) \times D_\alpha \times D_\alpha$ such that $0 = s_0 < s_1 < \cdots < s_j < \cdots < s_{i-1} < \sigma$ and conditions (i) through (iii) are satisfied for $1 \leqslant j \leqslant i-1$. By $\bar{h}_i$ we denote the supremum of numbers $h \in [0, \sigma - s_{i-1})$ such that

$$\begin{aligned}&V_\beta\big(s_{i-1}+h+\tau_0, T(h)\bar{w}_{i-1}+hB\bar{w}_{i-1}, T(h)\hat{w}_{i-1}+hB\hat{w}_{i-1}\big)\\&\quad - V_\beta(s_{i-1}+\tau_0,\bar{w}_{i-1},\hat{w}_{i-1})\\&\quad\leqslant \big(\omega V_\beta(s_{i-1}+\tau_0,\bar{w}_{i-1},\hat{w}_{i-1}) + \big(L(\beta)/4\big)(\bar{\eta}+\hat{\eta})\big)h.\end{aligned}$$

Since $\bar{h}_i > 0$ by condition (ii-2) of Theorem 2.3, there exists $h_i > 0$ such that $\bar{h}_i/2 < h_i < \sigma - s_{i-1}$ and

$$\begin{aligned}&\big(V_\beta\big(s_{i-1}+h_i+\tau_0, T(h_i)\bar{w}_{i-1}+h_iB\bar{w}_{i-1}, T(h_i)\hat{w}_{i-1}+h_iB\hat{w}_{i-1}\big)\\&\quad - V_\beta(s_{i-1}+\tau_0,\bar{w}_{i-1},\hat{w}_{i-1})\big)/h_i\\&\quad\leqslant \omega V_\beta(s_{i-1}+\tau_0,\bar{w}_{i-1},\hat{w}_{i-1}) + \big(L(\beta)/4\big)(\bar{\eta}+\hat{\eta}).\end{aligned}\tag{4.6}$$

We set $s_i = s_{i-1} + h_i$. We shall apply Proposition 3.4 to show the existence of $\bar{w}_i, \hat{w}_i \in D_\alpha$ satisfying conditions (i) and (ii) for $j = i$. To this end, we employ Lemma 3.1 with $v_0 = \bar{v}_0$, $w_0 = \bar{w}_0$, $\delta = \bar{\delta}$ and with $\bar{\varepsilon}$ replaced by $\bar{\eta}/2 + \bar{\varepsilon}$, so that

$$\big\|\bar{w}_{i-1} - T(s_{i-1}+\bar{\delta})\bar{v}_0\big\| \leqslant (s_{i-1}+\bar{\delta})(\overline{M} + \bar{\eta}/2 + \bar{\varepsilon}).$$

Since $\varphi(\bar{w}_{i-1}) \leqslant m^{\bar{\varepsilon}}(s_{i-1};\varphi(\bar{w}_0)) \leqslant m^{\bar{\varepsilon}}(s_{i-1}+\bar{\delta};\varphi(\bar{v}_0))$ and $h_i + (s_{i-1}+\bar{\delta}) < \sigma + \bar{\delta} \leqslant \bar{h}$, we can apply Proposition 3.4 with $v_0 = \bar{v}_0$, $\delta = s_{i-1}+\bar{\delta}$, $w_0 = \bar{w}_{i-1}$, $\sigma = h_i$ and with $\bar{\eta}$ and $\overline{M}$ replaced by $\bar{\eta}/8$ and $\overline{M} + \bar{\eta}/2$, respectively. Thus, we find $\bar{w}_i \in D_\alpha$ such that

$$\begin{aligned}&\big\|T(h_i)\bar{w}_{i-1} + h_iB\bar{w}_{i-1} - \bar{w}_i\big\| \leqslant h_i(\bar{\varepsilon}+\bar{\eta}/2),\\&\varphi(\bar{w}_i) \leqslant m^{\bar{\varepsilon}}\big(h_i;\varphi(\bar{w}_{i-1})\big).\end{aligned}\tag{4.7}$$

In the same way we find $\hat{w}_i \in D_\alpha$ satisfying the desired estimates. Condition (iii) follows from (V2), (4.6) and (4.7). To show that $\lim_{j\to\infty} s_j = \sigma$, assume to the contrary that $\lim_{j\to\infty} s_j = \bar{s} < \sigma$. Then, applying (ii) of Lemma 3.2 with $v_0 = \bar{v}_0$, $w_0 = \bar{w}_0$, $\delta = \bar{\delta}$ and with $\bar{\varepsilon}$ and $\bar{\eta}$ replaced by $\bar{\eta}/2 + \bar{\varepsilon}$ and $\bar{\eta}/8$, we observe that the sequence $\{\bar{w}_j\}$ in D_α converges in X to some $\bar{w} \in D_\alpha$. The convergence in X of the sequence $\{\hat{w}_j\}$ to some $\hat{w} \in D_\alpha$ is proved similarly. Since $\bar{s} < \sigma \leqslant \tau - \tau_0$, there exists $h \in (0, \sigma - \bar{s})$ such that

$$\begin{aligned}&\big(V_\beta\big(\bar{s}+h+\tau_0, T(h)\bar{w}+hB\bar{w}, T(h)\hat{w}+hB\hat{w}\big) - V_\beta(\bar{s}+\tau_0,\bar{w},\hat{w})\big)/h\\&\quad\leqslant \omega V_\beta(\bar{s}+\tau_0,\bar{w},\hat{w}) + \big(L(\beta)/8\big)(\bar{\eta}+\hat{\eta}).\end{aligned}$$

Set $\gamma_i = \bar{s} + h - s_{i-1}$ for $i \geqslant 1$. Then, since $s_{i-1} + \gamma_i + \tau_0 = \bar{s} + h + \tau_0$ for $i \geqslant 1$ and $\lim_{i\to\infty}\gamma_i = h$ we use condition (V2) to deduce that

$$\begin{aligned}&V_\beta\big(s_{i-1}+\gamma_i+\tau_0, T(\gamma_i)\bar{w}_{i-1}+\gamma_iB\bar{w}_{i-1}, T(\gamma_i)\hat{w}_{i-1}+\gamma_iB\hat{w}_{i-1}\big)\\&\quad\to V_\beta\big(\bar{s}+h+\tau_0, T(h)\bar{w}+hB\bar{w}, T(h)\hat{w}+hB\hat{w}\big)\end{aligned}$$

as $i \to \infty$. By condition (V1) we see that

$$V_\beta(s_{i-1} + \tau_0, \bar{w}_{i-1}, \hat{w}_{i-1}) \to V_\beta(\bar{s} + \tau_0, \bar{w}, \hat{w})$$

as $i \to \infty$. Therefore, we find a positive integer $i_0 \geqslant 1$ such that $\gamma_i \leqslant \bar{h}_i$ for $i \geqslant i_0$. This contradicts the fact that $\gamma_i \to h$ (> 0) and $\bar{h}_i \to 0$ as $i \to \infty$. Thus, we obtain the desired sequence $\{(s_j, \bar{w}_j, \hat{w}_j)\}_{j=0}^\infty$ in $[0, \sigma) \times D_\alpha \times D_\alpha$.

Applying Lemma 3.2 to the sequence $\{(s_j, \bar{w}_j, \hat{w}_j)\}_{j=0}^\infty$ with $v_0 = \bar{v}_0$, $w_0 = \bar{w}_0$, $\delta = \bar{\delta}$ and with $\bar{\varepsilon}$ and $\bar{\eta}$ replaced by $\bar{\varepsilon} + \bar{\eta}/2$ and $\bar{\eta}/8$, respectively, we find $\bar{z}_0, \hat{z}_0 \in D_\alpha$ such that $\bar{z}_0 = \lim_{j\to\infty} \bar{w}_j$, $\hat{z}_0 = \lim_{j\to\infty} \hat{w}_j$,

$$\left\|T(\sigma)\bar{w}_0 + \sigma B\bar{w}_0 - \bar{z}_0\right\| \leqslant \sigma(\bar{\varepsilon} + \bar{\eta}),$$
$$\left\|T(\sigma)\hat{w}_0 + \sigma B\hat{w}_0 - \hat{z}_0\right\| \leqslant \sigma(\hat{\varepsilon} + \hat{\eta}).$$

From (ii) and (iii) we deduce that $\varphi(\bar{w}_j) \leqslant m^{\bar{\varepsilon}}(s_j; \varphi(\bar{w}_0))$ and

$$V_\beta(s_j + \tau_0, \bar{w}_j, \hat{w}_j) \leqslant \exp(\omega s_j)\bigl(V_\beta(\tau_0, \bar{w}_0, \hat{w}_0) + s_j L(\beta)(\bar{\eta} + \hat{\eta} + \bar{\varepsilon} + \hat{\varepsilon})\bigr)$$

for $j \geqslant 1$, respectively. Letting $i \to \infty$ in the inequalities above, we have $\varphi(\bar{z}_0) \leqslant m^{\bar{\varepsilon}}(\sigma; \varphi(\bar{w}_0))$ and

$$V_\beta(\sigma + \tau_0, \bar{z}_0, \hat{z}_0) \leqslant \exp(\omega\sigma)\bigl(V_\beta(\tau_0, \bar{w}_0, \hat{w}_0) + \sigma L(\beta)(\bar{\eta} + \hat{\eta} + \bar{\varepsilon} + \hat{\varepsilon})\bigr),$$

by conditions (φ) and (V1). The proof of Proposition 4.1 is thus complete. □

Proposition 4.2. *Let $\alpha \in \mathbb{R}_+^n$ and $x_0 \in D_\alpha$. Let $\tau \geqslant \bar{\tau} > 0$, $R > 0$, $M_B > 0$ and $\lambda, \mu \in (0, 1/4)$ and suppose that*

$$\|Bx\| \leqslant M_B \quad \text{for } x \in U[x_0, R] \cap D_\alpha,$$
$$\bar{\tau}(M_B + 1) + \sup_{s\in[0,\bar{\tau}]} \left\|T(s)x_0 - x_0\right\| \leqslant R,$$
$$\text{for each } \varepsilon = \lambda, \mu,\ \bar{\tau} < \tau^\varepsilon\bigl(\varphi(x_0)\bigr) \text{ and } m^\varepsilon\bigl(s; \varphi(x_0)\bigr) \leqslant \alpha \quad \text{for } s \in [0, \bar{\tau}].$$

For each $\varepsilon = \lambda, \mu$, suppose that there exists a sequence $\{(t_j^\varepsilon, x_j^\varepsilon)\}_{j=0}^\infty$ in $[0, \bar{\tau}) \times D_\alpha$ satisfying $x_0^\varepsilon = x_0$ and the following conditions:

(i) $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_j^\varepsilon < \cdots < \bar{\tau}$.
(ii) $t_j^\varepsilon - t_{j-1}^\varepsilon \leqslant \varepsilon$ *for* $j \geqslant 1$.
(iii) $\|T(t_j^\varepsilon - t_{j-1}^\varepsilon)x_{j-1}^\varepsilon + (t_j^\varepsilon - t_{j-1}^\varepsilon)Bx_{j-1}^\varepsilon - x_j^\varepsilon\| \leqslant (\varepsilon/2)(t_j^\varepsilon - t_{j-1}^\varepsilon)$ *for* $j \geqslant 1$.
(iv) *If $x \in D_\alpha$ satisfies*

$$\left\|x - x_{j-1}^\varepsilon\right\| \leqslant \bigl(t_j^\varepsilon - t_{j-1}^\varepsilon\bigr)(M_B + 1) + \sup_{s\in[0, t_j^\varepsilon - t_{j-1}^\varepsilon]} \left\|T(s)x_{j-1}^\varepsilon - x_{j-1}^\varepsilon\right\|,$$

then $\|Bx - Bx_{j-1}^\varepsilon\| \leqslant \varepsilon/8$, *for* $j \geqslant 1$.
(v) $\sup_{s\in[0, t_j^\varepsilon - t_{j-1}^\varepsilon]} \|T(s)Bx_{j-1}^\varepsilon - Bx_{j-1}^\varepsilon\| \leqslant \varepsilon/8$ *for* $j \geqslant 1$.
(vi) $\varphi(x_j^\varepsilon) \leqslant m^\varepsilon(t_j^\varepsilon - t_{j-1}^\varepsilon; \varphi(x_{j-1}^\varepsilon))$ *for* $j \geqslant 1$.
(vii) $\lim_{j\to\infty} t_j^\varepsilon = \bar{\tau}$.

Set $P = \{t_i^\lambda;\ i = 0, 1, \ldots\} \cup \{t_j^\mu;\ j = 0, 1, \ldots\}$, and define $s_0 = 0$ and $s_k = \inf(P \setminus \{s_0, s_1, \ldots, s_{k-1}\})$ for $k \geqslant 1$. Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^\infty$ in $D_\alpha \times D_\alpha$ satisfying the following conditions:

(a) *If* $s_k = t_i^\lambda$, *then* $z_k^\lambda = x_i^\lambda$; *if* $s_k = t_j^\mu$, *then* $z_k^\mu = x_j^\mu$.
(b) *For each* $\varepsilon = \lambda, \mu$,

$$\sum_{j=q}^{k} \left\| T(s_j - s_{j-1}) z_{j-1}^\varepsilon + (s_j - s_{j-1}) B z_{j-1}^\varepsilon - z_j^\varepsilon \right\| \leqslant 2\varepsilon(s_k - s_{q-1}) + 3\varepsilon \sum_{t_i^\varepsilon \in \{s_q, \dots, s_k\}} \left(t_i^\varepsilon - t_{i-1}^\varepsilon\right) \tag{4.8}$$

for $k \geqslant 1$ *and* $1 \leqslant q \leqslant k$.
(c) *For each* $\varepsilon = \lambda, \mu$,

$$\varphi\left(z_k^\varepsilon\right) \leqslant m^\varepsilon\left(s_k - t_{i-1}^\varepsilon; \varphi\left(x_{i-1}^\varepsilon\right)\right)$$

for $k \geqslant 1$ *and* $i \geqslant 1$ *with* $t_{i-1}^\varepsilon < s_k \leqslant t_i^\varepsilon$.
(d) $V_\beta(s_k, z_k^\lambda, z_k^\mu) \leqslant \exp(\omega s_k)(2L(\beta)(\lambda + \mu)s_k + \eta_k(\lambda, \mu))$ *for* $k \geqslant 0$, *where*

$$\eta_k(\lambda, \mu) = 3L(\beta)\left(\lambda \sum_{t_i^\lambda \in \{s_1, \dots, s_k\}} \left(t_i^\lambda - t_{i-1}^\lambda\right) + \mu \sum_{t_j^\mu \in \{s_1, \dots, s_k\}} \left(t_j^\mu - t_{j-1}^\mu\right)\right).$$

Here β *and* ω *are constants depending only on the given vector* α.

Proof. We inductively construct the desired sequence. Set $z_0^\varepsilon = x_0$ for $\varepsilon = \lambda, \mu$. Let $l \geqslant 1$ and assume that a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^{l-1}$ in $D_\alpha \times D_\alpha$ can be chosen so that conditions (a) through (d) are satisfied. Then we want to find the desired pair $(z_l^\lambda, z_l^\mu) \in D_\alpha \times D_\alpha$ by applying Proposition 4.1. To this end, let i and j be the integers such that $t_{i-1}^\lambda < s_l \leqslant t_i^\lambda$ and $t_{j-1}^\mu < s_l \leqslant t_j^\mu$. Then we observe by the definition of $\{s_k\}$ that $t_{i-1}^\lambda \leqslant s_{l-1} < s_l \leqslant t_i^\lambda$ and $t_{j-1}^\mu \leqslant s_{l-1} < s_l \leqslant t_j^\mu$, and hence there exists an integer p such that $0 \leqslant p \leqslant l-1$ and $t_{i-1}^\lambda = s_p$. This fact together with (a) implies $x_{i-1}^\lambda = z_p^\lambda$. By (iii) of Lemma 3.1 with $\bar{h} = \sigma = \bar{\tau}$, $\bar{r} = R$, $\overline{M} = M_B$, $v_0 = w_0 = x_0$, $\delta = 0$ and $\bar{\varepsilon} = \lambda/2$, we have $\|Bx_{i-1}^\lambda\| \leqslant M_B$. By this fact and condition (iv) we see that

$$\|Bx\| \leqslant M_B + \lambda/8 \quad \text{for } x \in U\left[x_{i-1}^\lambda, \bar{r}_i^\lambda\right] \cap D_\alpha, \tag{4.9}$$

where

$$\bar{r}_i^\lambda := \left(t_i^\lambda - t_{i-1}^\lambda\right)(M_B + 1) + \sup_{s \in [0, t_i^\lambda - t_{i-1}^\lambda]} \left\| T(s) x_{i-1}^\lambda - x_{i-1}^\lambda \right\|.$$

Since $\varphi(x_{i-1}^\lambda) \leqslant m^\lambda(t_{i-1}^\lambda; \varphi(x_0))$ by condition (vi) with $j = 1, \dots, i-1$, we have, by (m1), (m2) and the assumption on $\bar{\tau}$,

$$t_i^\lambda - t_{i-1}^\lambda < \tau^\lambda\left(\varphi\left(x_{i-1}^\lambda\right)\right) \quad \text{and} \quad m^\lambda\left(s; \varphi\left(x_{i-1}^\lambda\right)\right) \leqslant \alpha \quad \text{for } s \in \left[0, t_i^\lambda - t_{i-1}^\lambda\right]. \tag{4.10}$$

We need to show that

$$\left\| z_{l-1}^\lambda - T(s_{l-1} - s_p) x_{i-1}^\lambda \right\| \leqslant (s_{l-1} - s_p)(M_B + \lambda/8 + 2\lambda), \tag{4.11}$$

$$\varphi\left(z_{l-1}^\lambda\right) \leqslant m^\lambda\left(s_{l-1} - t_{i-1}^\lambda; \varphi\left(x_{i-1}^\lambda\right)\right). \tag{4.12}$$

We have only to consider the case where $p < l-1$. Since the set $\{s_{p+1}, \dots, s_{l-1}\}$ has no elements t_i^λ, we infer from (b) that

$$\left\| T(s_j - s_{j-1}) z_{j-1}^\lambda + (s_j - s_{j-1}) B z_{j-1}^\lambda - z_j^\lambda \right\| \leqslant 2\lambda(s_j - s_{j-1}) \tag{4.13}$$

for $j = p+1, \dots, l-1$. By (4.9) and the definition of $\bar{r}$, the desired inequality (4.11) is obtained by noting that $s_p = t_{i-1}^{\lambda}$ and $z_p^{\lambda} = x_{i-1}^{\lambda}$ and applying Lemma 3.1 with $\bar{r} = \bar{r}_i^{\lambda}$, $\bar{h} = \sigma = t_i^{\lambda} - t_{i-1}^{\lambda}$, $v_0 = w_0 = x_{i-1}^{\lambda}$, $\bar{\varepsilon} = 2\lambda$, $\delta = 0$ and $\overline{M} = M_B + \lambda/8$ to the sequence $\{(s_{j+p} - t_{i-1}^{\lambda}, z_{j+p}^{\lambda})\}_{j=0}^{l-1-p}$ satisfying (4.13). Since $t_{i-1}^{\lambda} = s_p < s_{l-1} < s_l \leqslant t_i^{\lambda}$, the desired inequality (4.12) is nothing but the assumption (c) of induction argument.

By (4.9) through (4.12) (iv), (v) and the fact that $M_B + \lambda/8 + 3\lambda \leqslant 1$, assumptions (4.1) through (4.5) of Proposition 4.1 are satisfied with $\bar{v}_0 = x_{i-1}^{\lambda}$, $\bar{r} = \bar{r}_i^{\lambda}$, $\overline{M} = M_B + \lambda/8 + \lambda$, $\bar{\eta} = \lambda$, $\bar{h} = t_i^{\lambda} - t_{i-1}^{\lambda}$, $\bar{\varepsilon} = \lambda$, $\bar{\delta} = s_{l-1} - s_p = s_{l-1} - t_{i-1}^{\lambda}$ and $\bar{w}_0 = z_{l-1}^{\lambda}$. In a way similar to the above argument, we see that all the other assumptions of Proposition 4.1 are satisfied with $\hat{v}_0 = x_{j-1}^{\mu}$, $\widehat{M} = M_B + \mu/8 + \mu$, $\hat{\eta} = \mu$, $\hat{r} = (t_j^{\mu} - t_{j-1}^{\mu})(M_B + 1) + \sup_{s \in [0, t_j^{\mu} - t_{j-1}^{\mu}]} \|T(s)x_{j-1}^{\mu} - x_{j-1}^{\mu}\|$, $\hat{h} = t_j^{\mu} - t_{j-1}^{\mu}$, $\hat{\varepsilon} = \mu$, $\hat{\delta} = s_{l-1} - t_{j-1}^{\mu}$ and $\hat{w}_0 = z_{l-1}^{\mu}$. We therefore apply Proposition 4.1 with $\sigma = s_l - s_{l-1}$ and $\tau_0 = s_{l-1}$ to find $(y_l^{\lambda}, y_l^{\mu}) \in D_{\alpha} \times D_{\alpha}$ such that

$$
\begin{aligned}
&\left\|T(s_l - s_{l-1})z_{l-1}^{\lambda} + (s_l - s_{l-1})Bz_{l-1}^{\lambda} - y_l^{\lambda}\right\| \leqslant 2\lambda(s_l - s_{l-1}), &(4.14)\\
&\left\|T(s_l - s_{l-1})z_{l-1}^{\mu} + (s_l - s_{l-1})Bz_{l-1}^{\mu} - y_l^{\mu}\right\| \leqslant 2\mu(s_l - s_{l-1}), &\\
&V_{\beta}\big(s_l, y_l^{\lambda}, y_l^{\mu}\big) \leqslant \exp\big(\omega(s_l - s_{l-1})\big)\big(V_{\beta}\big(s_{l-1}, z_{l-1}^{\lambda}, z_{l-1}^{\mu}\big) &\\
&\qquad\qquad + 2(s_l - s_{l-1})L(\beta)(\lambda + \mu)\big), &(4.15)\\
&\varphi\big(y_l^{\lambda}\big) \leqslant m^{\lambda}\big(s_l - s_{l-1}; \varphi\big(z_{l-1}^{\lambda}\big)\big), &(4.16)\\
&\varphi\big(y_l^{\mu}\big) \leqslant m^{\mu}\big(s_l - s_{l-1}; \varphi\big(z_{l-1}^{\mu}\big)\big). &
\end{aligned}
$$

Now, we define a pair $(z_l^{\lambda}, z_l^{\mu}) \in D_{\alpha} \times D_{\alpha}$ by

$$
z_l^{\lambda} = \begin{cases} y_l^{\lambda} & \text{if } s_l < t_i^{\lambda}, \\ x_i^{\lambda} & \text{if } s_l = t_i^{\lambda} \end{cases} \quad \text{and} \quad z_l^{\mu} = \begin{cases} y_l^{\mu} & \text{if } s_l < t_j^{\mu}, \\ x_j^{\mu} & \text{if } s_l = t_j^{\mu}. \end{cases}
$$

By (4.9), (iv) and (v) we apply (i) of Lemma 3.2 with $v_0 = w_0 = x_{i-1}^{\lambda} = z_p^{\lambda}$, $\delta = 0$, $\bar{h} = t_i^{\lambda} - t_{i-1}^{\lambda}$, $\bar{r} = \bar{r}_i^{\lambda}$, $\overline{M} = M_B + \lambda/8$, $\bar{\eta} = \lambda/8$ and $\bar{\varepsilon} = 2\lambda$ to the sequence satisfying (4.13) and (4.14), so that

$$
\left\|T(s_l - s_p)z_p^{\lambda} + (s_l - s_p)Bz_p^{\lambda} - y_l^{\lambda}\right\| \leqslant (s_l - s_p)\big(2\lambda + (3/8)\lambda\big).
$$

If $s_l = t_i^{\lambda}$, then condition (iii) and the inequality above together imply that

$$
\left\|x_i^{\lambda} - y_l^{\lambda}\right\| \leqslant 3\lambda\big(t_i^{\lambda} - t_{i-1}^{\lambda}\big);
$$

hence

$$
\left\|z_l^{\lambda} - y_l^{\lambda}\right\| \leqslant 3\lambda \sum_{t_i^{\lambda} = s_l} \big(t_i^{\lambda} - t_{i-1}^{\lambda}\big). \tag{4.17}
$$

By using (4.14) and (4.17) it can be shown that condition (b) with $k = l$ holds. In the case where $t_{i-1}^{\lambda} < s_{l-1}$ and $s_l < t_i^{\lambda}$, the assumption of induction argument that $\varphi(z_{l-1}^{\lambda}) \leqslant m^{\lambda}(s_{l-1} - t_{i-1}^{\lambda}; \varphi(x_{i-1}^{\lambda}))$ combined with (4.16) implies that condition (c) holds for $k = l$. The desired claim in the other cases follows from condition (vi) and (4.16). Similarly to the derivation of (4.17) we have $\|z_l^{\mu} - y_l^{\mu}\| \leqslant 3\mu \sum_{t_j^{\mu} = s_l} (t_j^{\mu} - t_{j-1}^{\mu})$. From this inequality, (4.15) and (4.17) we deduce that

$$
V_{\beta}\big(s_l, z_l^{\lambda}, z_l^{\mu}\big) \leqslant 3L(\beta)\lambda \sum_{t_i^{\lambda} = s_l} \big(t_i^{\lambda} - t_{i-1}^{\lambda}\big) + 3L(\beta)\mu \sum_{t_j^{\mu} = s_l} \big(t_j^{\mu} - t_{j-1}^{\mu}\big)
$$

$$+ \exp\big(\omega(s_l - s_{l-1})\big)\big(V_\beta\big(s_{l-1}, z_{l-1}^\lambda, z_{l-1}^\mu\big) + 2(s_l - s_{l-1})L(\beta)(\lambda+\mu)\big).$$

Combining the inequality above and (d) with $k = l-1$, we observe that condition (d) is valid for $k = l$. The desired sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^\infty$ in $D_\alpha \times D_\alpha$ is thus obtained by induction. $\square$

We are now in a position to give the proof of implication (ii) $\Rightarrow$ (i) of Theorem 2.3.

Proof of Theorem 2.3 (ii) $\Rightarrow$ (i). If the (SP; x) has a global mild solution $u(t;x)$ for every $x \in D$, then the desired semigroup $\{S(t);\ t \geqslant 0\}$ on D is obtained by setting $S(t)x = u(t;x)$ for $x \in D$ and $t \geqslant 0$. By Propositions 2.5 and 2.6 we have only to prove the existence of a mild solution on certain interval for every $x \in D$. To do this, let $x_0 \in D$. Set $\alpha_i = \varphi_i(x_0) + 1$ for $i = 1, 2, \ldots, n$ and $\alpha = (\alpha_i)_{i=1}^n$. Notice that $x_0 \in D_\alpha$. By condition (B), there exist $R > 0$ and $M_B > 0$ satisfying $\|Bx\| \leqslant M_B$ for $x \in U[x_0, R] \cap D_\alpha$, and then we find $a > 0$ and $b > 0$ such that $m_i(s; \varphi(x_0)) \leqslant \varphi_i(x_0) + 1/2$ for $s \in [0, a]$ and $i = 1, 2, \ldots, n$ and such that $b(M_B + 1) + \sup_{s\in[0,b]} \|T(s)x_0 - x_0\| \leqslant R$. Set $\bar\tau = a \wedge b \wedge \tau$. Then we have

$$\bar\tau(M_B + 1) + \sup_{s\in[0,\bar\tau]} \big\|T(s)x_0 - x_0\big\| \leqslant R,$$

$$m_i\big(s; \varphi(x_0)\big) \leqslant \varphi_i(x_0) + 1/2 \quad \text{for } s \in [0, \bar\tau] \text{ and } i = 1, 2, \ldots, n.$$

We use condition (m3) to choose $\varepsilon_0 \in (0, 1/4)$ so that $\varepsilon \in (0, \varepsilon_0]$ implies that

$$m^\varepsilon\big(s; \varphi(x_0)\big) \leqslant \alpha \quad \text{for } s \in [0, \bar\tau], \quad \text{and} \quad \bar\tau < \tau^\varepsilon\big(\varphi(x_0)\big).$$

Then, Proposition 3.5 asserts that for each $\varepsilon \in (0, \varepsilon_0]$ there exists a sequence $\{(t_j^\varepsilon, x_j^\varepsilon)\}_{j=0}^\infty$ in $[0, \bar\tau) \times D_\alpha$ satisfying (i) through (viii) in Proposition 3.5. For each $\varepsilon \in (0, \varepsilon_0]$, we define a sequence $\{u^\varepsilon\}$ of step functions by $u^\varepsilon(t) = x_i^\varepsilon$ for $t \in [t_i^\varepsilon, t_{i+1}^\varepsilon)$ and $i = 0, 1, 2, \ldots$. Let $\lambda, \mu \in (0, \varepsilon_0]$, and let $\{s_k\}_{k=0}^\infty$ be the sequence constructed in Proposition 4.2. Then, applying Proposition 4.2, we find a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^\infty$ in $D_\alpha \times D_\alpha$ satisfying (a) through (d) of Proposition 4.2. In the following, let β and ω are constants appearing in condition (d) of Proposition 4.2.

Let $t \in [0, \bar\tau)$. We want to estimate $\|u^\lambda(t) - u^\mu(t)\|$. Let i, j, k be positive integers such that $t \in [s_{k-1}, s_k)$, $t_{i-1}^\lambda \leqslant s_{k-1} < s_k \leqslant t_i^\lambda$ and $t_{j-1}^\mu \leqslant s_{k-1} < s_k \leqslant t_j^\mu$. Then we have $u^\lambda(t) = x_{i-1}^\lambda$ and $u^\mu(t) = x_{j-1}^\mu$. In a way similar to the proof of (4.11) we have

$$\big\|z_{k-1}^\lambda - T\big(s_{k-1} - t_{i-1}^\lambda\big)x_{i-1}^\lambda\big\| \leqslant \big(s_{k-1} - t_{i-1}^\lambda\big)(M_B + \lambda/8 + 2\lambda).$$

Since this inequality and (v) of Proposition 3.5 imply that $\|z_{k-1}^\lambda - x_{i-1}^\lambda\| \leqslant \lambda$, we have $|V_\beta(s_{k-1}, x_{i-1}^\lambda, x_{j-1}^\mu) - V_\beta(s_{k-1}, z_{k-1}^\lambda, z_{k-1}^\mu)| \leqslant L(\beta)(\lambda + \mu)$ by condition (V2). Since $m(\beta)\|x_{i-1}^\lambda - x_{j-1}^\mu\| \leqslant V_\beta(s_{k-1}, x_{i-1}^\lambda, x_{j-1}^\mu)$ by condition (V3), we deduce from (d) of Proposition 4.2 that

$$m(\beta)\big\|u^\lambda(t) - u^\mu(t)\big\| \leqslant 5\exp(\omega\bar\tau)L(\beta)(\lambda+\mu)\bar\tau + L(\beta)(\lambda+\mu).$$

This implies the existence of a measurable function $u : [0, \bar\tau) \to X$ such that $\lim_{\lambda\downarrow 0} u^\lambda(t) = u(t)$ uniformly for $t \in [0, \bar\tau)$. From condition (vii) of Proposition 3.5 we deduce that $\varphi(u(t)) \leqslant m(t; \varphi(x_0)) \leqslant \alpha$ for $t \in [0, \bar\tau)$. By (iii) of Proposition 3.5 we have

$$\big\|T\big(t_i^\lambda - t_{j-1}^\lambda\big)x_{j-1}^\lambda + \big(t_j^\lambda - t_{j-1}^\lambda\big)T\big(t_i^\lambda - t_j^\lambda\big)Bx_{j-1}^\lambda - T\big(t_i^\lambda - t_j^\lambda\big)x_j^\lambda\big\| \leqslant (\varepsilon/2)\big(t_j^\lambda - t_{j-1}^\lambda\big)$$

for $1 \leqslant j \leqslant i$. Adding the inequalities above from $j = 1$ to i, we have

$$\left\| T\left(t_i^\lambda\right)x_0 + \sum_{j=1}^{i} \int_{t_{j-1}^\lambda}^{t_j^\lambda} T\left(t_i^\lambda - t_j^\lambda\right) B x_{j-1}^\lambda \, ds - x_i^\lambda \right\| \leqslant (\varepsilon/2) t_i^\lambda$$

for $i \geqslant 0$. Since $\|Bu^\lambda(t)\| \leqslant M_B$ for $t \in [0, \bar{\tau})$ and $\lim_{\lambda \downarrow 0} Bu^\lambda(t) = Bu(t)$ for $t \in [0, \bar{\tau})$, we have $Bu \in L^\infty(0, \bar{\tau}; X)$. We therefore apply the dominated convergence theorem to obtain

$$T(t)x_0 + \int_0^t T(t-s)Bu(s)\,ds = u(t)$$

for $t \in [0, \bar{\tau})$. The continuity of u on $[0, \bar{\tau})$ in X follows from the equality above. The proof of Theorem 2.3 is thus complete. □

5. An application to the complex Ginzburg–Landau equation

Let Ω be a general domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, where $1 \leqslant N \leqslant 4$. Let us consider the existence and uniqueness of global solutions to the mixed problem for the complex Ginzburg–Landau equation

$$\begin{cases} \frac{\partial u}{\partial t} - (\lambda + i\mu)\Delta u + (\kappa + i\nu)|u|^{q-2}u - \gamma u = 0 & \text{in } \Omega \times (0, \infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \end{cases} \tag{CGL}$$

where $\lambda > 0$, $\kappa > 0$, $\mu, \nu, \gamma \in \mathbb{R}$ and q is assumed to satisfy the condition

$$2 \leqslant q \leqslant 2 + 4/N. \tag{5.1}$$

In the case where Ω is bounded, Okazawa and Yokota [18] recently have studied the same problem without the restriction $1 \leqslant N \leqslant 4$, using their abstract result formulated by subdifferential operators.

Let $X = L^2(\Omega)$ and denote the norm and the inner product in X by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Then, the linear operator A in X defined by

$$Au = (\lambda + i\mu)\Delta u \quad \text{for } u \in D(A) := H_0^1(\Omega) \cap H^2(\Omega)$$

is the infinitesimal generator of a contractive (C_0) semigroup $\{T(t); t \geqslant 0\}$ on X. Since $1 \leqslant N \leqslant 4$ and $2 \leqslant q \leqslant 2 + 4/N$, we see that $H^1(\Omega) \subset L^{2(q-1)}(\Omega)$ and the inclusion is continuous. Define a nonlinear operator B_0 in X by

$$B_0 u = -(\kappa + i\nu)|u|^{q-2}u + \gamma u \quad \text{for } u \in D(B_0) := H^1(\Omega).$$

Then, we are in a position to state the following result.

Theorem 5.1. *There exists a semigroup $\{S(t);\ t \geqslant 0\}$ of locally Lipschitz operators on $L^2(\Omega)$ satisfying the following conditions*:

(i) *For each $\tau, r > 0$ there exists $M(\tau, r) > 0$ such that $\|S(t)u_0 - S(t)v_0\| \leqslant M(\tau, r)\|u_0 - v_0\|$ for $t \in [0, \tau]$ and $u_0, v_0 \in L^2(\Omega)$ with $\|u_0\| \leqslant r$, $\|v_0\| \leqslant r$.*
(ii) *$\|S(t)u_0\| \leqslant \exp(\gamma t)\|u_0\|$ for $t \geqslant 0$ and $u_0 \in L^2(\Omega)$.*

(iii) *$S(t)(H_0^1(\Omega)) \subset H_0^1(\Omega)$ for $t \geqslant 0$, $S(\cdot)u_0 : [0,\infty) \to H_0^1(\Omega)$ is continuous for every $u_0 \in H_0^1(\Omega)$, and $S(t)u_0 = T(t)u_0 + \int_0^t T(t-s)B_0S(s)u_0\,ds$ for $t \geqslant 0$ and $u_0 \in H_0^1(\Omega)$.*

(iv) *For each $u_0 \in H_0^1(\Omega)$, the* (CGL) *has a solution $S(t)u_0$ in the class $C([0,\infty); H_0^1(\Omega)) \cap C^1((0,\infty); L^2(\Omega))$.*

To prove Theorem 5.1, we define $D = H_0^1(\Omega) \cap H^2(\Omega)$ and apply Theorem 2.3 with A and the nonlinear operator B in X defined by $Bu = B_0u$ for $u \in D$. To prove condition (B), we employ the three functionals defined by

$$\varphi_0(u) = \|u\|^2 \quad \text{for } u \in L^2(\Omega),$$
$$\varphi_1(u) = \begin{cases} \exp((b/2\kappa)\|u\|^2)\|\nabla u\|^2 & \text{if } u \in H_0^1(\Omega), \\ \infty & \text{otherwise}, \end{cases}$$
$$\varphi_2(u) = \begin{cases} \exp((b/2\kappa)\|u\|^2)\|Au + Bu\|^2 & \text{if } u \in H_0^1(\Omega) \cap H^2(\Omega), \\ \infty & \text{otherwise}, \end{cases}$$

where $b > 0$ is yet to be determined. Let $\varphi(u) = (\varphi_0(u), \varphi_1(u), \varphi_2(u))$ for $u \in L^2(\Omega)$. Since

$$\big||\xi|^{q-2}\xi - |\eta|^{q-2}\eta\big| \leqslant (q-1)\int_0^1 \big|\theta\xi + (1-\theta)\eta\big|^{q-2}|\xi - \eta|\,d\theta \tag{5.2}$$

for $\xi, \eta \in \mathbb{C}$, we have

$$\|Bu - Bv\| \leqslant L_B\big(\|u\|_{L^{2(q-1)}} \vee \|v\|_{L^{2(q-1)}}\big)^{q-2}\|u - v\|_{L^{2(q-1)}} + |\gamma|\|u - v\| \tag{5.3}$$

for $u, v \in D$. By the Gagliardo–Nirenberg inequality

$$\|u\|_{L^r} \leqslant C\|u\|^{1-\sigma}\|u\|_{H^1}^{\sigma} \quad \text{for } u \in H^1(\Omega), \tag{5.4}$$

where $1 \leqslant r < \infty$ and $\sigma = N(1/2 - 1/r)$, we see by (5.1) that the topology of $L^2(\Omega)$ is stronger than that of $L^{2(q-1)}(\Omega)$ on each bounded set in $H^1(\Omega)$. This fact combined with the inequality (5.3) shows that condition (B) is satisfied. Since each revel set of φ is bounded in $H^1(\Omega)$, the closedness of the revel set of φ in X follows from the lower semicontinuity of the norm $\|\cdot\|$, the graph of A is weakly closed in $X \times X$ and the continuity of B in X on each bounded set in $H^1(\Omega)$.

We begin by showing that to each $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ there corresponds $h_0 > 0$ such that for each $h \in (0, h_0]$, the equation

$$(u_h - u_0)/h - (\lambda + i\mu)\Delta u_h + (\kappa + i\nu)|u_h|^{q-2}u_h - \gamma u_h = 0 \tag{5.5}$$

has a solution $u_h \in H_0^1(\Omega) \cap H^2(\Omega)$. To prove the above fact we need the following lemma.

Lemma 5.2.

(i) *The operator A satisfies the following conditions*:
 (i-1) $\|(I - hA)^{-1}v\|_{H^1} \leqslant (1 + Ch^{-1/2})\|v\|$ *for $h > 0$ and $v \in L^2(\Omega)$.*
 (i-2) $\lim_{h\downarrow 0}\|(I - hA)^{-1}v - v\|_{H^1} = 0$ *for any $v \in H_0^1(\Omega)$.*

(ii) *For each $r > 0$ there exists $L_B(r) > 0$ such that*

$$\|B_0u - B_0v\| \leqslant L_B(r)\|u - v\|_{H^1} \tag{5.6}$$

for $u, v \in H^1(\Omega)$ with $\|u\|_{H^1} \leqslant r$, $\|v\|_{H^1} \leqslant r$.

Proof. Let $\{T_1(t);\ t \geqslant 0\}$ be the analytic semigroup on X generated by the operator $A_1u := \Delta u$ for $u \in D$. Then it is known [22] that $\|(-A_1)^{1/2}T_1(t)\| \leqslant Ct^{-1/2}$ for $t > 0$. The operator $A_2u := i\mu\Delta u$ for $u \in D$ is the infinitesimal generator of a unitary group $\{T_2(t);\ t \in \mathbb{R}\}$ on X. Since $T(t) = T_1(\lambda t)T_2(t)$ for $t \geqslant 0$ (by the Trotter–Kato product formula) and $\|(-A_1)^{1/2}v\| = \|\nabla v\|$ for $v \in D((-A_1)^{1/2}) = H_0^1(\Omega)$ we have

$$\left\|T(t)v\right\|_{H^1} \leqslant \left(1 + Ct^{-1/2}\right)\|v\| \tag{5.7}$$

for all $t > 0$ and $v \in L^2(\Omega)$. Hence assertion (i-1) follows by the Laplace transform. Since $(-A_1)^{1/2}(I - hA)^{-1}v = (I - hA)^{-1}(-A_1)^{1/2}v$ for $v \in H_0^1(\Omega)$ and $h > 0$, the desired assertion (i-2) is true. Assertion (ii) is a direct consequence of the inequality (5.2). □

To show the above-mentioned claim concerning range condition, let $u_0 \in D$ and $h > 0$. Here it should be noticed that $h > 0$ will be chosen sufficiently small in the later argument. Let $E = \{v \in H^1(\Omega);\ \|v - u_0\|_{H^1} \leqslant 1\}$ and consider the mapping Φ from E into D defined by

$$\Phi v = (I - hA)^{-1}(u_0 + hB_0v) \quad \text{for } v \in E.$$

Then we want to show that Φ is a strictly contractive mapping from E into itself. To do this, let $r = \|u_0\|_{H^1} + 1$ and $v \in E$. Since

$$\Phi v - u_0 = \left((I - hA)^{-1}u_0 - u_0\right) + h(I - hA)^{-1}\left((B_0v - B_0u_0) + B_0u_0\right),$$

we have $\|\Phi v - u_0\|_{H^1} \leqslant \|(I - hA)^{-1}u_0 - u_0\|_{H^1} + (h + Ch^{1/2})(L_B(r) + \|B_0u_0\|)$ by assertions (i-1) and (ii) of Lemma 5.2. Assertion (i-2) asserts that the right-hand side vanishes as $h \downarrow 0$; hence there exists $h_0 > 0$ depending only on u_0 such that $h \in (0, h_0]$ implies that $\Phi(E) \subset E$. We employ assertions (i-1) and (ii) of Lemma 5.2 again to obtain

$$\|\Phi u - \Phi v\|_{H^1} \leqslant \left(h + Ch^{1/2}\right)\|B_0u - B_0v\| \leqslant L_B(r)\left(h + Ch^{1/2}\right)\|u - v\|_{H^1}$$

for any $u, v \in E$. This implies that Φ is strictly contractive on E, for every $h \in (0, h_0]$ such that $L_B(r)(h + Ch^{1/2}) < 1$. We apply the Banach–Picard fixed point theorem to find an element $u_h \in D$ such that $u_h = (I - hA)^{-1}(u_0 + hB_0u_h)$, for sufficiently small $h > 0$ depending only on u_0. Thus, the desired claim is proved.

To check condition (ii-3) in Theorem 2.3, let $h \in (0, h_0]$ be such that $1 - 2\gamma h > 0$. By (5.5) we have

$$2\,\mathrm{Re}\left\langle u_h, (u_h - u_0)/h\right\rangle + 2\lambda\|\nabla u_h\|^2 + 2\kappa\|u_h\|_{L^q}^q - 2\gamma\|u_h\|^2 = 0. \tag{5.8}$$

Since $2\,\mathrm{Re}\langle u, u - v\rangle \geqslant \|u\|^2 - \|v\|^2$ for $u, v \in L^2(\Omega)$, we find that

$$\left(\|u_h\|^2 - \|u_0\|^2\right)/h + 2\lambda\|\nabla u_h\|^2 + 2\kappa\|u_h\|_{L^q}^q - 2\gamma\|u_h\|^2 \leqslant 0, \tag{5.9}$$

which implies that the sequence $\{u_h\}$ is bounded in $L^2(\Omega)$ and

$$\|u_h\|^2 \leqslant (1 - 2h\gamma)^{-1}\|u_0\|^2. \tag{5.10}$$

By (5.9) and (5.10) we have $\limsup_{h\downarrow 0}(\varphi_0(u_h) - \varphi_0(u_0))/h \leqslant 2\gamma\varphi_0(u_0)$.

By (5.9) we have $h\|u_h\|_{L^q}^q \leqslant (2\kappa)^{-1}(\|u_0\|^2 + 2\gamma h\|u_h\|^2)$; hence

$$\limsup_{h\downarrow 0} h\left|\left\langle |u_h|^{q-2}u_h, \phi\right\rangle\right| \leqslant \limsup_{h\downarrow 0}\left(h\|u_h\|_{L^q}^q\right)^{(q-1)/q}h^{1/q}\|\phi\|_{L^q} = 0$$

for any $\phi \in C_0^\infty(\Omega)$. It follows from (5.5) that the sequence $\{u_h\}$ converges weakly to u_0 in $L^2(\Omega)$ as $h \downarrow 0$. Since $\limsup_{h\downarrow 0}\|u_h\|^2 \leqslant \|u_0\|^2$ by (5.10), we have $\lim_{h\downarrow 0} u_h = u_0$ in $L^2(\Omega)$.

Since $|\nabla(|u|^{q-2}u)| \leqslant (q-1)|u|^{q-2}|\nabla u|$ for $u \in H^1(\Omega)$, we have $|\langle -\Delta u_h, |u_h|^{q-2}u_h\rangle| \leqslant (q-1)\|u_h\|_{L^q}^{q-2}\|\nabla u_h\|_{L^q}^2$. By (5.4) with $r = q$, the right-hand side is bounded by $C_q\|u_h\|_{L^q}^{q-2}\|\nabla u_h\|^{2(1-\sigma)}\|u_h\|_{H^2}^{2\sigma}$. Since $\|u\|_{H^2}^2 \leqslant C(\|u\|^2 + \|\Delta u\|^2)$ for $u \in H_0^1(\Omega) \cap H^2(\Omega)$, we find that

$$\left|\langle -\Delta u_h, |u_h|^{q-2}u_h\rangle\right| \leqslant C_q\|u_h\|_{L^q}^{q-2}\|\nabla u_h\|^{2(1-\sigma)}\left(\|u_h\|^{2\sigma} + \|\Delta u_h\|^{2\sigma}\right),$$

where $\sigma = N(1/2 - 1/q)$. We take the inner product of (5.5) and $-\Delta u_h$, and use the inequality above. This yields that

$$\begin{aligned}&\left(\|\nabla u_h\|^2 - \|\nabla u_0\|^2\right)/h + 2\lambda\|\Delta u_h\|^2\\&\quad\leqslant C_q\|u_h\|_{L^q}^{q-2}\|\nabla u_h\|^{2(1-\sigma)}\left(\|u_h\|^{2\sigma} + \|\Delta u_h\|^{2\sigma}\right) + 2\gamma\|\nabla u_h\|^2.\end{aligned}$$

Since $\xi\eta \leqslant C_{\lambda,\sigma}\xi^{1/(1-\sigma)} + \lambda\eta^{1/\sigma}$ for $\xi, \eta \geqslant 0$ and $(q-2)/(1-\sigma) \leqslant q$ by (5.1), we find that

$$\left(\|\nabla u_h\|^2 - \|\nabla u_0\|^2\right)/h \leqslant \left(a + b\|u_h\|_{L^q}^q\right)\|\nabla u_h\|^2 + \lambda\|u_h\|^2 \tag{5.11}$$

for some nonnegative constants a, b. Since $\lim_{h\downarrow 0} u_h = u_0$ in $L^2(\Omega)$, we have $\limsup_{h\downarrow 0} h\|u_h\|_{Lq}^q \leqslant \limsup_{h\downarrow 0}(2\kappa)^{-1}(\|u_0\|^2 - (1-2h\gamma)\|u_h\|^2) = 0$ by (5.9). This fact together with (5.11) implies that $\limsup_{h\downarrow 0}\|\nabla u_h\|^2 \leqslant \|\nabla u_0\|^2$. Since the sequence $\{\nabla u_h\}$ converges weakly to ∇u_0 in $L^2(\Omega)^N$ as $h \downarrow 0$, we see that the sequence $\{\nabla u_h\}$ converges to ∇u_0 in $L^2(\Omega)^N$ as $h \downarrow 0$. It follows that $\lim_{h\downarrow 0} u_h = u_0$ in $H^1(\Omega)$. Since the functional $u \to \exp((b/2\kappa)\|u\|^2)$ is convex, we have

$$\begin{aligned}&h^{-1}\left(\exp\left((b/2\kappa)\|u_h\|^2\right) - \exp\left((b/2\kappa)\|u_0\|^2\right)\right)\\&\quad\leqslant (b/2\kappa)\exp\left((b/2\kappa)\|u_h\|^2\right)2\operatorname{Re}\langle u_h, (u_h - u_0)/h\rangle.\end{aligned} \tag{5.12}$$

By (5.8), (5.11) and the above inequality we find that

$$\begin{aligned}\limsup_{h\downarrow 0}\left(\varphi_1(u_h) - \varphi_1(u_0)\right)/h &\leqslant \left(a + (b/\kappa)\gamma\varphi_0(u_0)\right)\varphi_1(u_0)\\&\quad + \lambda\exp\left((b/2\kappa)\phi_0(u_0)\right)\phi_0(u_0).\end{aligned} \tag{5.13}$$

Here we have used the fact that $\lim_{h\downarrow 0} u_h = u_0$ in $L^q(\Omega)$ (by the Sobolev imbedding theorem $H^1(\Omega) \subset L^q(\Omega)$).

Let $v_h = Au_h + Bu_h$ and $v_0 = Au_0 + Bu_0$. Then we have $v_h = (u_h - u_0)/h$ and

$$(v_h - v_0)/h - (\lambda + i\mu)\Delta v_h + (\kappa + i\nu)\left(|u_h|^{q-2}u_h - |u_0|^{q-2}u_0\right)/h - \gamma v_h = 0. \tag{5.14}$$

Taking the inner product of (5.14) and v_h, and using (5.2) we find that

$$\left(\|v_h\|^2 - \|v_0\|^2\right)/h + 2\lambda\|\nabla v_h\|^2 \leqslant C_q\left(\|u_h\|_{L^q} \vee \|u_0\|_{L^q}\right)^{q-2}\|v_h\|_{L^q}^2 + 2\gamma\|v_h\|^2.$$

By (5.4) with $r = q$ we have, similarly to the derivation of (5.11),

$$\left(\|v_h\|^2 - \|v_0\|^2\right)/h \leqslant \left(a + b\left(\|u_h\|_{L^q} \vee \|u_0\|_{L^q}\right)^q\right)\|v_h\|^2, \tag{5.15}$$

or $\|v_h\|^2 \leqslant (1 - (a + b(\|u_h\|_{L^q} \vee \|u_0\|_{L^q})^q)h)^{-1}\|v_0\|^2$. By this inequality, (5.8) and (5.12) we have

$$\begin{aligned}\left(\varphi_2(u_h) - \varphi_2(u_0)\right)/h &\leqslant \left(\exp\left((b/2\kappa)\|u_h\|^2\right)(b/2\kappa)\left(-2\kappa\|u_h\|_{L^q}^q + 2\gamma\|u_h\|^2\right)\right.\\&\quad \left. + \exp\left((b/2\kappa)\|u_0\|^2\right)\left(a + b\left(\|u_h\|_{L^q} \vee \|u_0\|_{L^q}\right)^q\right)\right)^+\|v_h\|^2,\end{aligned}$$

where the symbol $(\cdot)^+$ is defined by $(s)^+ = s \vee 0$ for $s \in \mathbb{R}$, and the right-hand side is bounded by $\exp((b/2\kappa)\|u_0\|^2)(a + (b/\kappa)\gamma\|u_0\|^2)\|v_0\|^2$ as $h \downarrow 0$. Since $u_h = (I - hA)^{-1}(u_0 + hBu_h)$, we have

$$\big(T(h)u_0 + hBu_0 - u_h\big)/h = \big(\big(T(h)u_0 - u_0\big)/h - \big((I - hA)^{-1}u_0 - u_0\big)/h\big) \\ + \big(Bu_0 - (I - hA)^{-1}Bu_h\big),$$

and the right-hand side vanishes as $h \downarrow 0$ because $u_0 \in D(A)$ and $\lim_{h\downarrow 0} Bu_h = Bu_0$ by condition (B). We therefore conclude that condition (ii-3) is satisfied with the comparison function $g(r) = (g_0(r), g_1(r), g_2(r))$ defined by

$$\begin{aligned} g_0(r) &= 2\gamma r_0, \\ g_1(r) &= \lambda \exp\big((b/2\kappa)r_0\big)r_0 + \big(a + (b/\kappa)\gamma r_0\big)r_1, \\ g_2(r) &= \big(a + (b/\kappa)\gamma r_0\big)r_2 \end{aligned}$$

for $r = (r_0, r_1, r_2) \in \mathbb{R}^3_+$.

To check condition (ii-1) in Theorem 2.3, let $\tau > 0$. For each $t \in [0, \tau]$ and $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3_+$, we define a functional V_α on $[0, \tau] \times L^2(\Omega) \times L^2(\Omega)$ by

$$V_\alpha(t, u, v) = \exp\big((b/2\kappa)\big(\big(\|u\| \wedge \sqrt{\alpha_0}\big)^2 + \big(\|v\| \wedge \sqrt{\alpha_0}\big)^2\big)\big)\big(\|u - v\| \wedge \big(2\sqrt{\alpha_0}\big)\big)$$

for $u, v \in L^2(\Omega)$. Notice that $u \in H^1_0(\Omega) \cap H^2(\Omega)$ and $\|u\| \leqslant \sqrt{\alpha_0}$ for $u \in D_\alpha$. We then see that condition (V3) is satisfied with $m(\alpha) = 1$ and $M(\alpha) = \exp((b/\kappa)\alpha_0)$. Since $|\xi \wedge \eta - (\bar{\xi} \wedge \bar{\eta})| \leqslant |\xi - \bar{\xi}| + |\eta - \bar{\eta}|$ for $\xi, \bar{\xi}, \eta, \bar{\eta} \in \mathbb{R}_+$ and

$$\begin{aligned} &\big|(d/d\theta)\exp\big((b/2\kappa)\big(\theta|\xi|^2 + (1 - \theta)|\eta|^2\big)\big)\big| \\ &\quad \leqslant \exp\big((b/2\kappa)\big(|\xi|^2 \vee |\eta|^2\big)\big)(b/2\kappa)\big(|\xi| + |\eta|\big)|\xi - \eta| \end{aligned}$$

for $\xi, \eta \in \mathbb{R}_+$ and $\theta \in (0, 1)$, we see that condition (V2) is satisfied with $L(\alpha) = \exp((b/\kappa)\alpha_0)((b/\kappa)\sqrt{2\alpha_0} + 1)$.

To check condition (ii-2) of Theorem 2.3, let $u_0, v_0 \in D_\alpha$ and set

$$\beta = \big(e^{2\gamma\tau}\alpha_0 + 1, \alpha_1, \alpha_2\big). \tag{5.16}$$

Then, we see that $\|u_0\| \leqslant \sqrt{\beta_0}$, where β_0 is the first component of β, and the solution u_h to Eq. (5.5) satisfies $\|u_h\| \leqslant \sqrt{\beta_0}$, since $\lim_{h\downarrow 0} u_h = u_0$ in $L^2(\Omega)$. Let w_h be the difference between u_h and the corresponding one v_h with v_0 in place of u_0, and let $w_0 = u_0 - v_0$. In a way similar to the derivation of (5.15), we have

$$\big(\|w_h\|^2 - \|w_0\|^2\big)/h \leqslant \big(a + b\big(\|u_h\|^q_{L^q} + \|v_h\|^q_{L^q}\big)\big)\|w_h\|^2.$$

This inequality together with (5.8) implies that

$$\begin{aligned} &\limsup_{h\downarrow 0}\big(V_\beta(t + h, u_h, v_h)^2 - V_\beta(t, u_0, v_0)^2\big)/h \\ &\quad \leqslant \big(a + 2(b/\kappa)\gamma\big(\|u_0\|^2 + \|v_0\|^2\big)\big)V_\beta(t, u_0, v_0)^2 \end{aligned} \tag{5.17}$$

for $t \in [0, \tau)$. This means that condition (ii-2) is satisfied, since V_β satisfies condition (V2) and $\lim_{h\downarrow 0}\|T(h)u_0 + hBu_0 - u_h\|/h = 0$.

By Theorem 2.3, there exists a semigroup $\{S(t);\ t \geqslant 0\}$ of locally Lipschitz operators on D with respect to the functional φ such that $S(t)u_0 = T(t)u_0 + \int_0^t T(t-s)BS(s)u_0\,ds$ for $u_0 \in D$ and $t \geqslant 0$. By (5.17), we see that for each $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3_+$ there exists $\omega(\alpha_0) \geqslant 0$ such that

$$V_\beta\big(t, S(t)u_0, S(t)v_0\big) \leqslant e^{\omega(\alpha_0)t} V_\beta(0, u_0, v_0)$$

for $t \in [0, \tau]$ and $u_0, v_0 \in D_\alpha$, where β is the vector defined by (5.16). By the growth condition we have $\varphi_0(S(t)u_0) \leqslant e^{2\gamma t}\varphi_0(u_0)$ for $t \geqslant 0$ and $u_0 \in D$. By the definition of V_β and φ_0, the above facts imply that $\|S(t)u_0 - S(t)v_0\| \leqslant \exp(\omega(\alpha_0)\tau + (b/\kappa)\alpha_0)\|u_0 - v_0\|$ for $t \in [0, \tau]$ and $u_0, v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ with $\|u_0\|^2 \leqslant \alpha_0$ and $\|v_0\|^2 \leqslant \alpha_0$ and that $\|S(t)u_0\| \leqslant \exp(\gamma t)\|u_0\|$ for $t \geqslant 0$ and $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. The unique extension of $S(t)$ to $L^2(\Omega)$ is the desired one.

To prove (iii), let $r_0, \tau_0 > 0$ and $u_0, \hat{u}_0 \in D$ be such that $\|u_0\|_{H^1} \leqslant r_0$ and $\|\hat{u}_0\|_{H^1} \leqslant r_0$. By the growth condition, there exists $r > 0$ such that $\|S(t)v\|_{H^1} \leqslant r$ for $t \in [0, \tau_0]$ and $v \in D$ with $\|v\|_{H^1} \leqslant r_0$. Since $S(t)v$ is a mild solution for each $v \in D$, we find by (5.6) and (5.7)

$$\begin{aligned}&\big\|S(t)u_0 - S(t)\hat{u}_0\big\|_{H^1}\\ &\quad\leqslant \|u_0 - \hat{u}_0\|_{H^1} + \int_0^t \big(1 + C(t-s)^{-1/2}\big)L_B(r)\big\|S(s)u_0 - S(s)\hat{u}_0\big\|_{H^1}\,ds\end{aligned}$$

for $t \in [0, \tau_0]$. An application of [8, Lemma 7.1.1] yields that $\|S(t)u_0 - S(t)\hat{u}_0\|_{H^1} \leqslant L(\tau_0, r_0)\|u_0 - \hat{u}_0\|_{H^1}$ for $t \in [0, \tau_0]$. Since D is dense in $H_0^1(\Omega)$, assertion (iii) follows from the inequality above.

Once it is shown that for each $u_0 \in H_0^1(\Omega)$, the function $B_0S(\cdot)u_0$ is locally Hölder continuous on $(0, \infty)$ in $L^2(\Omega)$, assertion (iv) is a direct consequence of [21, Corollary 4.3.3]. Now, let $0 < \varepsilon < \tau_0$ and $u_0 \in H_0^1(\Omega)$, and write

$$\begin{aligned}&(-A_1)^{1/2}\big(S(t)u_0 - S(\hat{t})u_0\big)\\ &\quad= \big(T(t-\hat{t}) - I\big)T(\hat{t})(-A_1)^{1/2}u_0 + \int_{\hat{t}}^t (-A_1)^{1/2}T(t-s)B_0S(s)u_0\,ds\\ &\qquad+ \int_0^{\hat{t}} \big(T(t-\hat{t}) - I\big)(-A_1)^{1/2}T(\hat{t}-s)B_0S(s)u_0\,ds\end{aligned} \tag{5.18}$$

for $t, \hat{t} \in [\varepsilon, \tau_0]$ with $\hat{t} < t$. Let $\eta \in (0, 1)$. Since $\|(-A_1)^{1-\eta}T_1(s)\| \leqslant Cs^{\eta-1}$ for $s > 0$ and $T(s)v - v = -(\lambda + i\mu)\int_0^s (-A_1)^{1-\eta}T_1(\lambda\sigma)T_2(\sigma)(-A_1)^\eta v\,d\sigma$ for $s \geqslant 0$ and $v \in D((-A_1)^\eta)$, we have

$$\big\|T(s)v - v\big\| \leqslant Cs^\eta\big\|(-A_1)^\eta v\big\| \tag{5.19}$$

for $s \geqslant 0$ and $v \in D((-A_1)^\eta)$. Here we have used the fact that $\{T_2(t);\ t \geqslant 0\}$ is a unitary group on $L^2(\Omega)$. Let $\delta \in (0, 1/2)$. By using (5.19) we estimate (5.18) to find

$$\begin{aligned}&\big\|(-A_1)^{1/2}\big(S(t)u_0 - S(\hat{t})u_0\big)\big\|\\ &\quad\leqslant C(t-\hat{t})^{1/2}\varepsilon^{-1/2}\|u_0\|_{H^1} + C\int_{\hat{t}}^t (t-s)^{-1/2}\,ds\,\big\|B_0S(\cdot)u_0\big\|_{C([0,\tau_0])}\end{aligned}$$

$$+ C(t-\hat{t})^{1/2-\delta} \int_0^{\hat{t}} (\hat{t}-s)^{\delta-1}\, ds \left\| B_0 S(\cdot)u_0 \right\|_{C([0,\tau_0])}$$
$$\leqslant C(t-\hat{t})^{1/2-\delta} \tag{5.20}$$

for $t, \hat{t} \in [\varepsilon, \tau_0]$ with $\hat{t} < t$. In a way similar to the derivation of (5.20) we have $\|S(t)u_0 - S(\hat{t})u_0\| \leqslant C(t-\hat{t})^{1/2}$ for $\varepsilon \leqslant \hat{t} < t \leqslant \tau_0$. This inequality combined with (5.20) implies that $S(\cdot)u_0 \in C^{1/2-\delta}([\varepsilon, \tau_0]; H_0^1(\Omega))$. By (5.6) we see that $BS(\cdot)u_0 \in C^{1/2-\delta}([\varepsilon, \tau_0]; L^2(\Omega))$. We conclude that $S(\cdot)u_0 \in C^1((0,\infty); L^2(\Omega))$ and $S(t)u_0$ satisfies the (CGL).

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